

~~Relativistic~~

UNIT - 2 1
MODERN PHYSICS & QUANTUM MECHANICS

MODERN PHYSICS

Questions

- 1) State and explain de-Broglie hypothesis
OR
Explain dual nature of matter
- 2) Derive an expression for wave length of an accelerated electron through a p. d of V volts.
- 3) What are matter waves? Give their properties
List out the properties of matter waves.
- 4) Explain phase velocity and group velocity
- 5) Define phase velocity and group velocity and hence derive the relation between them.
- 6) Define phase velocity and group velocity, Show that group velocity is equal to particle velocity
- 7) Derive the relation between phase velocity, group velocity and velocity of light
OR
Show that for a wave packet $v_p v_g = c^2$
- 8) Obtain the expression for de-Broglie wave length using group velocity
OR
Using the concept of group velocity, derive an expression for de-Broglie wavelength.

Wave Particle Dualism (OR) Wave Particle duality

The properties of light such as interference, diffraction and polarisation are explained by wave nature of light.

The photoelectric effect, Compton effect, Energy distribution of black body radiation are explained by particle nature of light.

It means light sometimes behaves as a wave and sometimes as behaves as a particle. Thus we ~~conclude~~ concluded that light exhibits dual nature. This ~~dual~~ ^{combination} of wave and particle properties ~~of light~~ is called as a wave-particle dualism. * Here it should be remembered that light can not exhibit both nature simultaneously. This dual nature concept is observed in material particles also, so that all particles exhibit both wave and particle properties. de-Broglie hypothesis.

Q.P Wave nature of particles (OR) —
In 1924 de-Broglie suggested that the dual nature of light extended to material particles such as electrons, protons, neutrons etc. Similar to light radiation matter has also dual nature. This dual nature of matter is known as de-Broglie hypothesis. It states that "Every moving particle ^{is} always associated with a wave". The wave associated with moving particles are known as matter waves or de Broglie waves. Thus a moving particle is associated with both wave properties and particle properties. The same can be expressed in mathematical equation.

Expression for de-Broglie wave length —

According to Planck's quantum theory, the energy of a photon is given by

$$E = h\nu = \frac{hc}{\lambda} \quad \text{--- (1)}$$

* This concept proposed by de-Broglie.

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According to Einstein Energy-mass relation, the energy of a photon is given by

$$E = mc^2 \quad \text{--- (2)}$$

Combining the two equations we get

$$\frac{hc}{\lambda} = mc^2$$

$$\frac{hc}{mc^2} = \lambda \quad \therefore \lambda = \frac{h}{mc}$$

This is de-Broglie wave-equation for a photon. de-Broglie extended this idea to material particles. Thus if a particle of mass m , velocity v , its momentum mv , then the wave length associated with this particle is given by

$$\lambda = \frac{h}{mv} = \frac{h}{p}$$

This equation is called de-Broglie wave length of matter waves.

Particle nature



{ wave representation
of a particle.

particle ~~Matter wave~~ wave nature of ~~matter~~ particles

de-Broglie wave length for a particle in terms

The de-Broglie waves are moving with a velocity v . Then the K.E of a moving particle is given by

$$E = \frac{1}{2} mv^2$$

$$\times \text{ and } \div \text{ by } m \quad E = \frac{m^2 v^2}{2m} = \frac{p^2}{2m} \quad \therefore p = \sqrt{2mE}$$

$$\text{Then } \lambda = \frac{h}{mv} = \frac{h}{p} = \frac{h}{\sqrt{2mE}}$$

This equation gives the de-Broglie wave length in terms of K.E

Q.P. de-Broglie wave length of an accelerated electron:-

Consider an electron of mass m and charge ' e ' is accelerated through a potential difference of V volt.

Then the work done on the electron by the electric field $= eV$ — (1)
(where V be the voltage applied to the electron)

This work done is equal to the gain of K.E of the electron

$$E = \frac{1}{2}mv^2 \quad \text{--- (2)}$$

Evaluating (1) and (2) we get

$$eV = \frac{1}{2}mv^2$$

$$2eV = mv^2$$

Multiply both sides by m we get

$$2meV = m^2v^2$$

$$2meV = (mv)^2$$

$$\sqrt{2meV} = mv$$

by de-Broglie wave equation

$$\lambda = \frac{h}{mv} \quad \text{--- (3)}$$

Substituting this mv in equation (3) we get

$$\lambda = \frac{h}{\sqrt{2meV}}$$

Substituting the values of h , m and e we get

$$\lambda = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 1.6 \times 10^{19} \times V}} = \frac{12.25 \text{ \AA}}{\sqrt{V}}$$

The de-Broglie wave length decreases when accelerating potential increases.

Matter waves

The waves associated with matter is known as matter waves (OR)

Every moving particle has associated with waves are called matter waves.

Properties of Matter waves

- ① Matter waves are associated with a moving particle
- ② For a particle at rest $v=0$ $\therefore \lambda = \infty$
i.e. particle at rest do not exhibit wave nature
- ③ The velocity of the matter wave is not constant it depends on the velocity of the material particle
- ④ Matter waves are associated with charged as well as uncharged particles. Hence matter waves are not electromagnetic waves.
- ⑤ Matter waves are called probability waves
- ⑥ Matter wave consists of group of waves
- ⑦ Matter waves can not be observed.
- ⑧ waves associated with macroscopic particles can not be measured.
- ⑨ $\lambda \propto \frac{1}{m}$ The wave length of matter wave inversely proportional to the mass of the particle: The larger the mass shorter will be the λ

[1] Find the de-Broglie wave length of an electron moving with one tenth of the velocity of light,

$$v = \frac{1}{10} \times c = 0.1 \times 3 \times 10^8$$

$$\lambda = \frac{h}{mv} = \frac{6.625 \times 10^{-34}}{9.11 \times 10^{-31} \times 3 \times 10^7}$$

$$v = 3 \times 10^7 \text{ m/s} \quad \lambda = 2.424 \times 10^{-11} \text{ m}$$

- ⑩ Matter waves are the waves that are associated with only micro particles of matter in motion.

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[2] Find the de-Broglie wave length of a bullet of mass 5 gm moving with a velocity 20 km/h

$$m = 5 \text{ gm} = 5 \times 10^{-3} \text{ Kg}$$

$$\lambda = \frac{h}{mv}$$

$$v = \frac{20 \times 1000 \text{ m}}{60 \times 60} = 5.556 \text{ m/s} \quad \lambda = \frac{6.625 \times 10^{-34}}{5 \times 10^{-3} \times 5.556} = 2.38 \times 10^{-32} \text{ m}$$

[3] Find the wave length associated with an electron having a kinetic energy of 100 eV

$$\begin{aligned} \text{Energy of the electron} = E &= 100 \text{ eV} \\ &= 100 \times 1.6 \times 10^{-19} \\ &= 1.6 \times 10^{-17} \text{ J} \end{aligned}$$

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 1.6 \times 10^{-17}}} = 1.227 \times 10^{-10} \text{ m}$$

[4] Find the de-Broglie wave length associated with neutron of mass $1.674 \times 10^{-27} \text{ Kg}$ with one tenth part of the velocity of light.

$$\lambda = \frac{h}{mv} \quad v = \frac{c}{10} \quad \text{Then } \lambda = 1.32 \times 10^{-14} \text{ m}$$

[5] A particle of mass $940 \text{ Mev}/c^2$ has kinetic energy 0.5 KeV. Find its de-Broglie wave length.

$$\begin{aligned} \text{Kinetic energy of the particle} = E &= 0.5 \text{ KeV} \\ &= 0.5 \times 10^3 \times 1.6 \times 10^{-19} \text{ J} = \dots \end{aligned}$$

$$\begin{aligned} \text{Mass of the particle} = m &= 940 \times 10^6 \text{ eV}/c^2 \\ m &= 940 \times 10^6 \times 1.6 \times 10^{-19} \text{ J}/c^2 \\ m &= \frac{940 \times 10^6 \times 1.6 \times 10^{-19}}{(3 \times 10^8)^2} = \dots \end{aligned}$$

$$\lambda = \frac{h}{\sqrt{2mE}} = 1.28 \times 10^{-12} \text{ m}$$

This λ_{obj} is shorter than EM radiation, this cannot be measured by any method. So the object does not have waves associated with it.

6] Find the de-Broglie wave length of an electron accelerated through a potential difference of 182 volt and object of mass 1 kg moving with a speed of 1 m/s. Compare the results and comment.

$$\lambda_e = \frac{h}{\sqrt{2meV}} = 0.91 \text{ \AA} \quad \lambda_{obj} = \frac{h}{mv} = 6.625 \times 10^{-24} \text{ \AA}$$

This wavelength is in atomic size and measurable. Comparison and comment: mass increasing the λ goes on decreasing, \therefore it is impossible to measure this wavelength.

7] Determine the de Broglie wave length associated with an electron having kinetic energy equal to 1 MeV.

8] Calculate the de-Broglie wave length associated with an electron of energy 1.5 eV

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 9.11 \times 10^{-31} \times 1.5 \times 10^{-19}}} = 1 \text{ nm or } 10 \text{ \AA}$$

Matter waves and their properties :-

The waves associated with matter is known as matter waves.

Properties of Matter waves

- 1) Matter waves are associated with a moving particle
- 2) For a particle at rest $v=0 \therefore \lambda = \infty$
i.e particle at rest do not exhibit wavenature
- 3) The velocity of the matter wave is not constant it depends on the velocity of the material particle
- 4) Matter waves are associated with charged as well as uncharged particles, Hence matter waves are not electromagnetic waves
- 5) Matter waves are called probability waves
- 6) Matter wave consists of group of waves
- 7) Matter waves cannot be observed.
- 8) Waves associated with macroscopic particles cannot be measured.

Phase Velocity of a de Broglie wave :-

Let v be the velocity of de-Broglie wave. ν is the frequency and λ is the wave length. Then $v = \nu \lambda$.
Let m be the mass of the particle associated with de-Broglie wave. Then energy of the particle

$$E = h\nu \quad \text{Then } \nu = \frac{E}{h}$$

$$v = \nu \lambda = \frac{E}{h} \cdot \lambda$$

Einstein's mass energy relation. The energy of the particle is given by $E = mc^2$

$$v = \nu \lambda = \frac{E}{h} \cdot \lambda = \frac{mc^2}{h} \cdot \lambda$$

$$\text{But } \lambda = \frac{h}{mv}$$

$$v = \frac{mc^2}{h} \times \frac{h}{mv} = \frac{c^2}{v}$$

$$\text{AS } v < c$$

$$\text{But } v > c$$

This indicates that phase velocity (v) of de Broglie wave is greater than velocity of light. According to theory of relativity the velocity of the wave is always less than the velocity of light. This contradicts the theory of relativity. Hence a material particle can not be equivalent to a single wave.

This difficulty is solved by assuming that each moving particle is associated with a group of waves. Therefore matter waves associated with moving particle is not a single wave. It consists of a group of waves or a wavepacket. In ^{was introduced.} such a motion both group and phase velocities are to be considered.

Phase Velocity (V_p) :-

The velocity of an individual wave in a wave packet is called phase velocity.

The velocity with which phase of the wave moves is called phase velocity.

It is defined as the ratio of angular frequency (ω) to the wave number (k)

$$\text{i.e. } V_p = \frac{\omega}{k} = \frac{\text{angular frequency}}{\text{wave number}}$$

This is phase velocity equation.

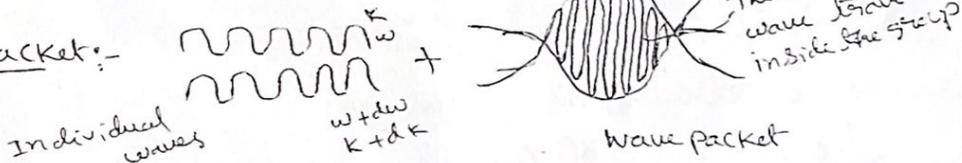
$$\text{Consider } \frac{\omega}{k} = \frac{2\pi\nu}{\frac{2\pi}{\lambda}} = \nu\lambda = V_p$$

The velocity with which a wave moves is called V_p and is equal to $\nu\lambda$ i.e. $V_p = \nu\lambda$

Group Velocity (V_g) :-

Due to the above reason de-Broglie waves are represented by a wave packet and hence we have group velocity associated with them.

Wave packet :-



Consider a group of two or more individual waves with a slight difference in the angular frequency and wave number as shown in figure. The superposition of individual waves takes place and a resultant wave as shown in figure is formed. The resultant wave is known as wave packet. It represents the motion of the particle. It is defined as a group of waves slightly different wave lengths.

The velocity with which the wave packet moves is called group velocity.

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It is defined as The ratio of Small difference in their frequencies to the wave numbers

Mathematically it is given by $v_g = \frac{d\omega}{dk}$

Differences between Phase velocity and group velocity :-

v_p	v_g
<ul style="list-style-type: none"> v_p is the velocity with which a wave moves v_p represents the velocity of the component waves of the wave packet. v_p can be greater than the velocity of light The phase velocity is given by the expression $v_p = \frac{\omega}{k}$ 	<ul style="list-style-type: none"> v_g is the velocity with which a wave packet moves v_g represents the velocity of the particle v_g is always less than the velocity of light The group velocity is given by the expression $v_g = \frac{d\omega}{dk}$

Relation between the group velocity and phase velocity :-

The phase velocity is given by

$$v_p = \frac{\omega}{k} \quad \text{or} \quad \omega = k \cdot v_p$$

The group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{d(k \cdot v_p)}{dk}$$

$$v_g = v_p \cdot 1 + k \frac{dv_p}{dk}$$

$$v_g = v_p + k \frac{dv_p}{d\lambda} \cdot \frac{d\lambda}{dk}$$

$$\text{But } k = \frac{2\pi}{\lambda}$$

$$\lambda = \frac{2\pi}{k}$$

diff w.r. to k we get

$$\frac{d\lambda}{dk} = -\frac{2\pi}{k^2}$$

$$V_g = V_p + \cancel{K} \frac{dV_p}{d\lambda} \left(-\frac{2\pi}{K^2} \right)$$

$$= V_p - \frac{2\pi}{K} \frac{dV_p}{d\lambda} \quad \text{But } K = \frac{2\pi}{\lambda}$$

$$= V_p - \frac{\frac{2\pi}{\lambda}}{\frac{2\pi}{\lambda}} \cdot \frac{dV_p}{d\lambda}$$

$$V_g = V_p - \lambda \frac{dV_p}{d\lambda} \quad \text{This is the relation between } V_g \text{ and } V_p.$$

This shows that the group velocity may be equal to, less than or more than the phase velocity of a wave depending upon the variation of phase velocity with respect to the wave length.

Relation between Group velocity and particle velocity :-

we have group velocity equation

OP.

$$V_g = \frac{d\omega}{dk} \quad \text{--- (1)}$$

$$\text{But } \omega = 2\pi\nu \quad h\nu = E \quad \nu = \frac{E}{h}$$

$$\omega = \frac{2\pi E}{h} \quad \text{--- (2)}$$

Differentiating

$$d\omega = \left(\frac{2\pi}{h} \right) dE \quad \text{--- (3)}$$

$$\text{Also we have } k = \frac{2\pi}{\lambda} = \frac{2\pi p}{h} \quad \text{--- (4)} \quad \text{But } \lambda = \frac{h}{p}$$

Differentiating

$$dk = \left(\frac{2\pi}{h} \right) dp \quad \text{--- (5)}$$

Dividing Equation (3) by equation (5) we get

$$\frac{d\omega}{dk} = \frac{dE}{dp} \quad \text{--- (6)}$$

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We also know that

$$E = \frac{1}{2}mv^2 = \frac{m^2v^2}{2m} = \frac{p^2}{2m}$$

$$dE = \frac{2p dp}{2m} = \frac{p}{m} dp$$

$$\frac{dE}{dp} = \frac{p}{m} \quad \text{--- (7)}$$

$$\frac{dE}{dp} = \frac{mv_{\text{particle}}}{m} = v_{\text{particle}}$$

$$\frac{dW}{dk} = v_{\text{particle}}$$

$$v_g = v_{\text{particle}}$$

Hence the group velocity of the wave packet is the same as the particle velocity.

Relation between Phase velocity, Group velocity and velocity of light :-

We know that $v_p = \frac{W}{K} = \frac{2\pi f}{\frac{2\pi}{\lambda}} = \frac{2\pi E}{h} = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v}$

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But we know that $v = v_g$

$$v_p = \frac{c^2}{v_g} \quad \text{or} \quad \boxed{v_p v_g = c^2}$$

Expression for de-Broglie wave length using group velocity

According to de-Broglie, a particle in motion associated with group of waves.

∴ The group velocity of the wave packet is given by

$$v_g = \frac{dW}{dk} \quad \text{--- (1)}$$

We know that

$$W = 2\pi f \quad \text{Differentiating} \quad dW = 2\pi df$$

$$K = \frac{2\pi}{\lambda} \quad \text{Differentiating} \quad dK = 2\pi d\left(\frac{1}{\lambda}\right)$$

Substituting these in equation (1) we get

$$v_g = \frac{d\omega}{dk} = \frac{2\pi d\nu}{2\pi d(1/\lambda)} = \frac{d\nu}{d(1/\lambda)}$$

$$d(1/\lambda) = \frac{d\nu}{v_g} \quad \text{--- (2)}$$

The Total energy of the particle is given by

$$E = \frac{1}{2}mv^2 + V \quad \text{--- (3)}$$

where $V =$ P.E is constant

But for matter waves $E = h\nu$

where ν is the freq of the wave associated with particle

$$\therefore h\nu = \frac{1}{2}mv^2 + V \quad \text{--- (4)}$$

Differentiating equation (4) we get

$$h d\nu = \frac{1}{2}m 2v dv$$

$$d\nu = \frac{mv dv}{h}$$

Substituting the value of $d\nu$ in equation (2) we get

$$d(1/\lambda) = \frac{d\nu}{v_g} = \frac{\frac{m}{h} v dv}{v_g} \quad \text{But } v_g = v_{\text{particle}}$$

$$d(1/\lambda) = \frac{m}{h} dv$$

Integrating we get $\int d(1/\lambda) = \int (\frac{m}{h}) dv$

$$1/\lambda = (\frac{m}{h})v + C$$

Assuming the constant of integration is zero

$$\therefore \text{we get } 1/\lambda = \frac{mv}{h}$$

$$\boxed{\lambda = \frac{h}{mv}}$$

This is deBroglie equation.

[1] A particle of mass $0.65 \text{ Mev}/c^2$ has a kinetic energy of 80 eV . Find its de-Broglie wavelength, group velocity and phase velocity of its de-Broglie waves.

Mass of the particle

$$m = 0.65 \text{ Mev}/c^2 = \frac{0.65 \times 10^6 \times 1.6 \times 10^{-19}}{(3 \times 10^8)^2} = 1.156 \times 10^{-30} \text{ kg}$$

de-Broglie wave length

$$\lambda = \frac{h}{\sqrt{2mE}} = \frac{6.625 \times 10^{-34}}{\sqrt{2 \times 1.156 \times 10^{-30} \times 80 \times 1.6 \times 10^{-19}}} = 1.218 \times 10^{-10} \text{ m} = 1.218 \text{ \AA}$$

$$\lambda = \frac{h}{mv} \quad v_g = v \quad v_g = v = \frac{h}{m\lambda} = \frac{6.625 \times 10^{-34}}{1.156 \times 10^{-30} \times 1.218 \times 10^{-10}} = 4.7 \times 10^6 \text{ m/s}$$

$$v_p \cdot v_g = c^2 \quad \therefore v_p = \frac{c^2}{v_g} = \frac{(3 \times 10^8)^2}{4.7 \times 10^6} = 1.91 \times 10^{10} \text{ m/s}$$

[2] Find the phase and group velocities of an electron whose de-Broglie wave length 1.2 \AA .

QR $\lambda = \frac{h}{mv} \quad v_g = v_{\text{particle}}$

$$v_g = \frac{h}{m\lambda} = \frac{6.625 \times 10^{-34}}{9.11 \times 10^{-31} \times 1.2 \times 10^{-10}} = 6.06 \times 10^6 \text{ m/s}$$

$$v_p = \frac{c^2}{v_g} = \frac{(3 \times 10^8)^2}{6.06 \times 10^6} = 1.485 \times 10^{10} \text{ m/s}$$

[3] Find the phase and group velocity of an electron whose de-Broglie wave length is 1.5 \AA .

$$v_g = v = \frac{h}{m\lambda} = 4.85 \times 10^6 \text{ m/s}$$

$$v_p = \frac{c^2}{v_g} = 1.85 \times 10^{10} \text{ m/sec.}$$