Unit-5

Z-transforms

- Z transform is an extended part of discrete time fourier transform.
- discrete time fourier transform can be applied only to stable system but Z-transforms can be applied or they can be use to calculate the functioning of unstable systems as well.
- Z transform is discrete counter part of Laplace transform.
- the primary use of Z transform is to study the system characteristic and derivation of computational structures for implementation of discrete time systems on computers.

* Defination of Z-transforms:-

Z-transform of $x(n)$ is denoted by $X(z)$. It is defined as.

$$\boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}} \quad -①$$

where z is a complex variable given by $z = re^{j\Omega}$.

where r is the magnitude, Ω = angle of z.

Substituting z in equation ① we have.

$$X(re^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\,(re^{j\Omega})^{-n}$$

$$X(re^{j}) = \sum_{n=-\infty}^{\infty} [x(n)\, r^{-n}]\, e^{-j\Omega n} \quad -②$$

from the equation (2) we see that

$X(re^{j\Omega})$ is the fourier transform of the sequence $x(n).r^{-n}$ i.e $X(re^{j\Omega}) = F\{x(n).r^{-n}\}$.

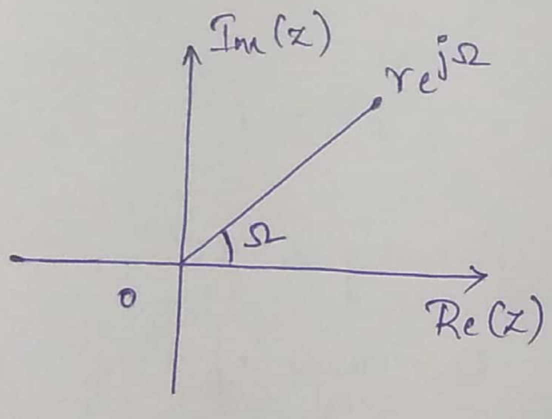
- the exponential weighting factor r^{-n} may be decaying or growing with increasing n, depending on whether 'r' is greater than or less than unity.

when $r = 1 \Rightarrow |z| = 1$.

$$\therefore X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n} = F\{x(n)\}$$

$$\Rightarrow X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}}.$$

ploting in Zplane. $z = re^{j\Omega}$ is located at a distance 'r' from origin and angle Ω relative to real axis.



$\therefore$ the relationship b/w $x(n)$ and $X[z]$ can be represented as

$$x(n) \xleftrightarrow{ZT} X[z]$$

* Types of z-transform :-

1. Bilateral Z-transform: Z transform defines both +ve & -ve sides of the Zplane they are called as Bilateral or both sided Z-transform.

2. Unilateral or one sided z-transform:-

It is defined as $X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$.

here summation starts from $n=0$ to ∞ ∴ only +ve axis of Zplane.

* Region of Convergence (ROC):-

- ROC is the region where Z transform converges.
- from the defination of Z transform it is clear that Z trasform is an infinite power series.
- this series is not convergent for all values of Z.

1. Finite Duration Sequence:-

a. Right-Sided Sequence:-

- A right-sided sequence is one for which $x(n) = 0$ for all $n \leq n_0$, where n_0 is +ve or -ve, but finite.
- If $n_0 \geq 0$, the resulting sequence $x(n)$ is a positive time sequence.
- For a Casual finite sequence ROC is entire Zplane except for $Z=0$

Eg $x(n) = (\underset{\uparrow}{1}, 2, 2, 1)$

By defination $X(Z) = \sum_{n=0}^{\infty} x(n) Z^{-n}$ (+ve sequence)

$$= \sum_{n=0}^{3} x(n) Z^{-n}.$$

$$= x(0) Z^{-0} + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3}$$

$= 1 + 2z^{-1} + 2z^{-2} + z^{-3}$

$= 1 + \frac{2}{z} + \frac{2}{z^2} + \frac{1}{z^3}$

∴ from above expression $x(z)$ is finite for all the values of z except $z = 0$. Hence ROC is entire z plane except $z = 0$ ∴ mathematically it can be written as.

$$\boxed{ROC : |z| > 0}$$

b. Left sided Sequence :-

- Sequence $x(n)$ is one for which $x(n) = 0$ for all $n > n_0$ where n_0 is positive or negative, but finite.
- If $n_0 \leq 0$, the resulting sequence $x(n)$ is called anti casual or negative sequence.
- for such kind of sequence ROC is entire z-plane except for $z = \infty$.

Eg $x(n) = (1, 1, 2, \underset{\uparrow}{2})$

By defination $x(z) = \sum_{n=-\infty}^{0} x(n) z^{-n}$.

$$\Rightarrow x(z) = \sum_{n=-3}^{0} x(n) . z^{-n}$$

$= x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0)$

$= z^{3} + z^{2} + 2z + 2.$

the expression $x(z)$ becomes infinite at $z = \infty$ hence ROC is entire z plane except for $z = \infty$

$$\boxed{ROC : |z| < \infty}$$

c. Double-Sided Sequence :-

- A Signal that has finite duration in both the +ve & -ve side is known as double-sided Sequence.
- In this case, ROC is the entire z-plane except for. points at $z=0$ and $z=\infty$.

Eg 1. $x(n) = (2, 1, \underset{\uparrow}{1}, 2)$

By defination $X(z) = \sum_{n=\infty}^{-\infty} x(n) z^{-n}$.

$$\Rightarrow X(z) = \sum_{n=-2}^{1} x(n) z^{-n}.$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1}$$

$$= 2z^{2} + z^{1} + 1 + 2z^{-1}$$

$\therefore$ ROC : $0 < |z| < \infty$.

X(z) becomes infinity at $z=0$ and $z=\infty$

$\Rightarrow$ ROC is the entire z-plane except for $z=0$ & $z=\infty$.

2. ROC of infinite Duration Sequence :-

a. Positive time exponential Sequence :-

A positive time exponential Sequence is defined by.

$$x(n) = a^{n} u(n).$$

$\therefore$ By z transfor defination

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^{n} u(n) z^{-n}$$

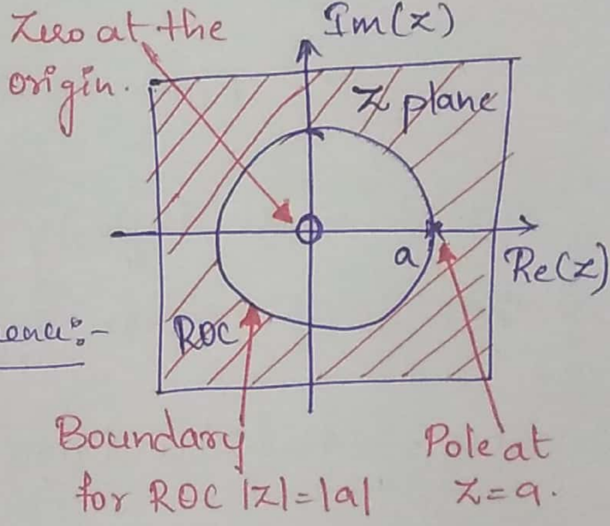
$$\Rightarrow X(z) = \sum_{n=0}^{\infty} a^n z^{-n} \quad [u(n) = 0 \text{ for } n < 0]$$

$$= \sum_{n=0}^{\infty} (a z^{-1})^n.$$

We know that $\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}$; $|\alpha| < 1$.

$$\Rightarrow X(z) = \frac{1}{1 - a z^{-1}} = \frac{z}{z-a}. \quad \text{ROC}: |z| > |a|$$

The above result converges if $|a z^{-1}| < 1$ which is equal to $|a| < |z|$. Values of z for which $X(z) = 0$ are called Zeros of $X(z)$, while the value of z for which $X(z) \to \infty$ are called the poles of $X(z)$



- poles are indicated as x
  Zeros are indicated by 0.

b. Negative time exponential Sequence:-

Sequence is defined by.

$$x(n) = -b^n u(-n-1)$$

We know that $u(-n-1) = \begin{cases} 1, & n \le -1 \\ 0, & n > -1. \end{cases}$

By z transformation defination

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$x(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n}.$$

$$= -\sum_{n=-\infty}^{-1} (b/z)^n.$$

Substituting $n = -m$ in the above summation, we get

$$x(z) = -\sum_{m=\infty}^{1} (z/b)^m.$$

$$= -\left[\sum_{m=1}^{\infty} (z/b)^m + 1 - 1\right]$$

$$= -\left[\sum_{m=1}^{\infty} (z/b)^m + (z/b)^0 - 1\right]$$

$$= -\left[\sum_{m=0}^{\infty} (z/b)^m - 1\right]$$

$$= 1 - \sum_{m=0}^{\infty} (z/b)^m.$$

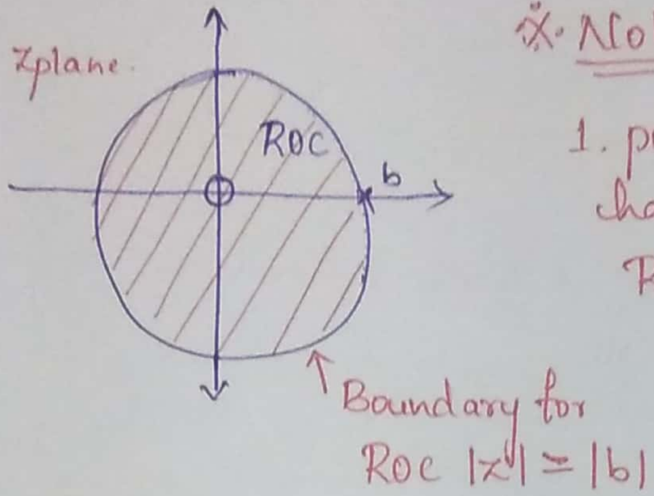
We know that $\displaystyle\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.$

$$\therefore \; x(z) = 1 - \frac{1}{1-(z/b)}$$

$\therefore$ Result Converges at $|z/b| < 1 = |z| < |b|$

ROC : $|z| < |b|$

ROC and pole-zero plot for negative time sequence is



※ **Note :-**

1. positive time sequence has z transform with ROC exterior to the circle $|z| = |a|$ $(|z| > |a|)$
2. negative time sequence has z transform with ROC interior to the circle $|z| = |b|$ $(|z| < |b|)$

* Double sided exponential sequence :-

double sided sequence is the sum of positive and negative exponential sequence.

i.e $x(n) = a^n u(n) - b^n u(-n-1)$

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$\therefore X(z) = \sum_{n=-\infty}^{\infty} [a^n u(n) - b^n u(-n-1)] z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \left[\frac{1}{1-az^{-1}}\right] + \left[1 - \frac{1}{1-(z/b)}\right]$$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

ROC: $|z| > |a| \cap |z| < |b|$

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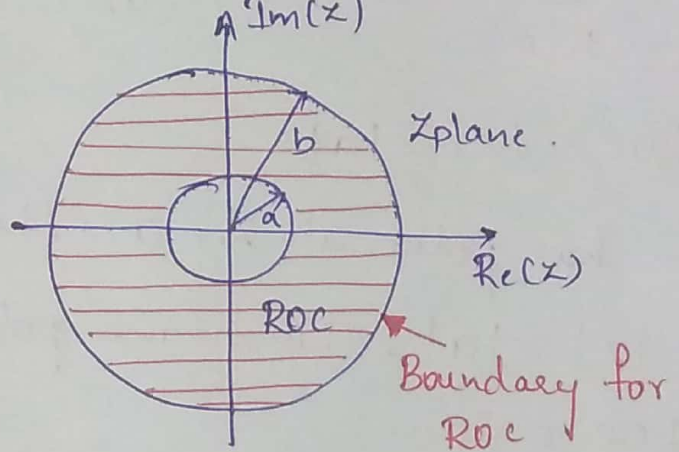
- the first power series converges if $|az^{-1}| < 1$. or $|z| > |a|$
- the second power series converges if $|(z/b)| < 1$ or $|z| < |b|$
- where as $X(z)$ is the sum of positive and negative time sequence with ROC equal to the intersection of the two region of convergence.



* **Properties of ROC :-**

We assume $X(z)$ is a rational function of z.

Property 1 : "ROC for a finite duration sequence includes entire Z-plane, except $|z| = 0$ and $|z| = \infty$"

Proof : $x(n) = \{1, 2, \underset{\uparrow}{1}, 2\}$

By defination of z transform

$$X(z) = 1z^{2} + 2z^{1} + 1 + 2z^{-1}$$

$$= z^{2} + 2z^{1} + 1 + \frac{2}{z}.$$

$X(z) = \infty$ for $z = 0$, & $z = \infty$.

Hence proved.

Property 2: ROC doesnot Contain any poles.

Proof: We have calculate z transform of $a^n u(n)$

$$X(Z) = \sum_{n=0}^{\infty} a^n z^{-n} \text{ [By defination]}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1-az^{-1}} = \frac{z}{z-a}.$$

ROC : $|z| > |a|$

this function has poles at $z = a$. Note that ROC is $|z| > |a|$ ∴ this means poles donot lie inside ROC region.

Property 3: "ROC is the ring in the Z-plane Centered about Origin"

Proof: Consider. $a^n u(n) \xleftrightarrow{ZT} \frac{z}{z-a}$, ROC $|z| > |a|$

or $-b^n u(-n-1) \xleftrightarrow{ZT} \frac{z}{z-b}$; ROC : $|z| < |b|$

Here we can observe that when $z=0$ the value obtained is 0 ∴ $|z|$ is always a circular region centered around origin.

Property 4: ROC of a casual Sequence (Right hand side Sequence) is of the form $|z| > r$.

Proof: Right hand Side Sequence = $a^n u(n)$

we know that is ROC: $|z| > |a|$

thus ROC of right hand sided sequence is of the form $|z| > r$. where r is the radius of the circle.

Property 5: ROC of left sided sequence is of the form $|z| < r$.

Proof: left sided sequence $= -b^n u(-n-1)$.

we know its ROC is $|z| < |b|$

this is of the form $|z| < r$ where r is the radius of the circle.

Property 6: ROC of two sided sequence is the concentric ring in z-plane.

Proof:- Double sided sequence $= a^n u(n) - b^n u(-n-1)$

we know its ROC is $|a| < |z| < |b|$

$\therefore$ ROC is the intersection of two regions of convergence hence it is concentric rings.

Property 7: If $x(n)$ is a finite casual sequence, then its ROC is entire z-plane except $z=0$.

Proof: $x(n) = \{1, 2, 3\}$ z-transforms will be

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{2} x(n) z^{-n} = x(0) + x(1) z^{-1} + x(2) z^{-2}.$$

$$= 1 + 2z^{-1} + 3z^{-2}.$$

hence the sequence converges in entire zplane except for $z=0$.

Property 8: the ROC of a stable LTI system contains unit circle in the Z-plane.

Property 9: The ROC is a connected region.

Proof: the convergence of the sequence exists over certain area, rather than discrete points.

Hence ROC is connected Region.

1. Determine the Z-transform of following sequence.

a. $x_1(n) = \{1, 2, 3, 4, 5, 0, 7\}$

By defination of Z transform.

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) Z^{-n}.$$

$$= \sum_{n=0}^{6} x(n) Z^{-n}.$$

$$= x(0) + x(1) Z^{-1} + x(2) Z^{-2} + x(3) Z^{-3} + x(4) Z^{-4} + x(5) Z^{-5} + x(6) Z^{-6}$$

$$= 1 + 2Z^{-1} + 3Z^{-2} + 4Z^{-3} + 5Z^{-4} + 0.Z^{-5} + 7Z^{-6}$$

$$\therefore X_1(Z) = 1 + \frac{2}{Z} + \frac{3}{Z^2} + \frac{4}{Z^3} + \frac{5}{Z^4} + \frac{7}{Z^6}.$$

$X_1(Z) = \infty$ if $Z = 0$

$\therefore X_1(Z)$ is convergent for all values of Z. except for $Z = 0$.

$\Rightarrow$ ROC: Entire Z plane except for $Z = 0$.

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b. $x_2(n) = \{1, 2, 3, \underset{\uparrow}{4}, 5, 0, 7\}$

By Defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$X_2(z) = \sum_{n=-3}^{3} x(n) z^{-n}.$$

$\Rightarrow x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$.

$$X_2(z) = 1z^3 + 2z^2 + 3z + 4 + 5z^{-1} + z^{-2} \cdot 0 + 7z^{-3}$$

$$= z^3 + 2z^2 + 3z + 4 + \frac{5}{z} + \frac{7}{z^3}$$

$X_2(z) = \infty$ for $z = 0$, $z = \infty$.

$\therefore$ ROC is entire z plane except for $z = 0$, $z = \infty$.

c. $x(n) = \{5, 2, \underset{\uparrow}{-2}, 1, 1, -3\}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=-2}^{3} x(n) z^{-n}.$$

$$= x(-2) z^{+2} + x(-1) z^{+1} + x(0) + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 5z^{2} + 2z - 2 + 1z^{-1} + 1z^{-2} + (-3) z^{-3}$$

$$= 5z^2 + 2z - 2 + \frac{1}{z} + \frac{1}{z^2} - \frac{3}{z^3}$$

$\therefore$ each term in $X(z)$ is finite and Consequently $X(z)$ will Converge for entire z plane except for $z = 0$, $z = \infty$.

2. find the z transform of $\delta(n)$

$$\delta(n) = \begin{cases} 1 & ; n = 0 \\ 0 & ; n \neq 0. \end{cases}$$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} \delta(n) z^{-n}.$$

$$= 1 . z^{-0} = 1.$$

this is a fixed value for any z. Hence ROC is entire z plane.

3. find the z transform of unit step sequence $u(n)$.

Unit step $u(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{for } n < 0 \end{cases}$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} u(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} 1 . z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^{n}.$$

$$\boxed{\sum_{n=0}^{\infty} \alpha^{n} = \frac{1}{1-\alpha}.}$$

$$= \frac{1}{1 - z^{-1}} = \frac{z}{z-1}.$$

$|z| > 1. \rightarrow$ ROC of $X(z)$

$|z|$ represents a circle in z plane where radius $= 1$.

"a unit circle" in z plane.

(4) For the signal $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$, find the Z-transform and ROC.

given: $x(n) = 7(1/3)^n u(n) - 6(1/2)^n u(n)$.

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$X(z) = \sum_{n=-\infty}^{\infty} \left[7(1/3)^n u(n) - 6(1/2)^n u(n)\right] z^{-n}.$$

$$= \sum_{n=0}^{\infty} 7(1/3)^n z^{-n} - \sum_{n=0}^{\infty} 6(1/2)^n z^{-n}.$$

$$= 7\sum_{n=0}^{\infty} (1/3 z^{-1})^n - 6\sum_{n=0}^{\infty} (1/2 z^{-1})^n.$$

$$= 7 \cdot \frac{1}{1-1/3 z^{-1}} - 6 \cdot \frac{1}{1 - 1/2 z^{-1}}$$

$$\boxed{\sum_{n=0}^{\infty} \alpha^n = \frac{1}{1-\alpha}.}$$

$$= \frac{7(1-1/2 z^{-1}) - 6(1-1/3 z^{-1})}{(1-1/3 z^{-1})(1-1/2 z^{-1})}$$

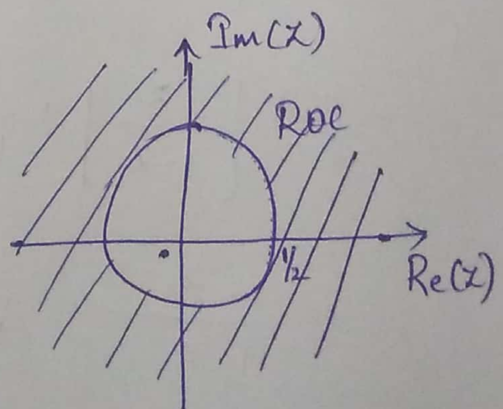
$$= \frac{1 - 3/2 z^{-1}}{(1-1/3 z^{-1})(1-1/2 z^{-1})}$$

the expression of $X(z)$ contains two ~~and~~ sums.

$\therefore |1/3 z^{-1}| < 1$ & $|1/2 z^{-1}| < 1$

$|1/3| < |z|$ & $|1/2| < |z|$

ROC: $|z| > |1/2|$



5. Determine the Z-transform of $x(n) = -u[-n-1] + (1/2)^n u(n)$ (16)
find the ROC and pole-zero location of $X(z)$ in the Z-plane.

By defination $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$$X(z) = \sum_{n=-\infty}^{\infty} [-u(-n-1) + (1/2)^n u(n)] z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} [-u(-n-1)] z^{-n} + \sum_{n=-\infty}^{\infty} (1/2)^n u(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} (-1) z^{-n} + \sum_{n=0}^{\infty} (1/2)^n z^{-n}$$

$$= (-1) \sum_{n=1}^{\infty} z^n + \sum_{n=0}^{\infty} (1/2 z^{-1})^n$$

$$= (-1) \left[\sum_{n=1}^{\infty} z^n + \cancel{1} - 1 \right] + \sum_{n=0}^{\infty} (1/2 z^{-1})^n$$

z^0

$$= (-1) \left[\sum_{n=0}^{\infty} z^n - 1 \right] + \frac{1}{1 - 1/2 z^{-1}}$$

$$= (-1) \left[\frac{1}{1-z} - 1 \right] + \frac{1}{1 - 1/2 z^{-1}}$$

$$= (-1) \left[\frac{z}{1-z} \right] + \frac{1}{1 - 1/2 z^{-1}}$$

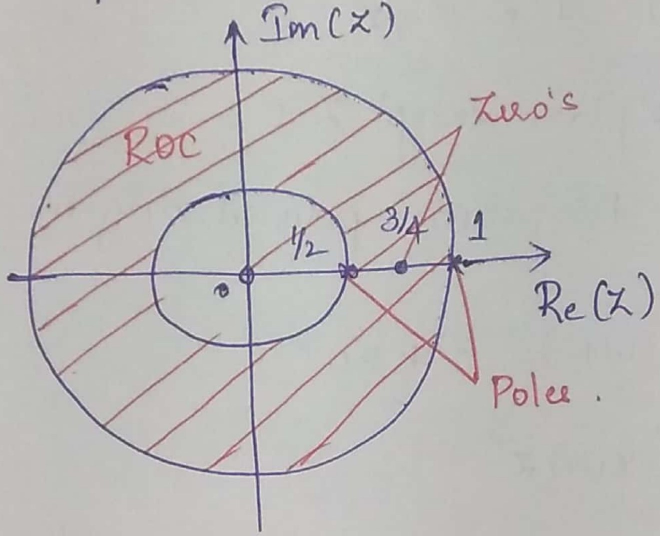
$$= \frac{-z}{1-z} + \frac{1}{1 - 1/2 z^{-1}}$$

$$= \frac{z}{(z-1)} + \frac{z}{(z-1/2)}$$

$$= \frac{2z^2 - 3/2 z}{(z-1)(z-1/2)} = \frac{z(2z - 3/2)}{(z-1)(z-1/2)}$$

$\therefore |z| < 1 \qquad |1/2 z^{-1}| < 1 \Rightarrow |1/2| < |z|$

$\therefore$ ROC $= |1/2| < |z| < 1$.



6. Determine the z transform and ROC for the following time signals. Sketch the ROC, poles and zero's in the z plane.

1. $x(n) = \delta(n)$

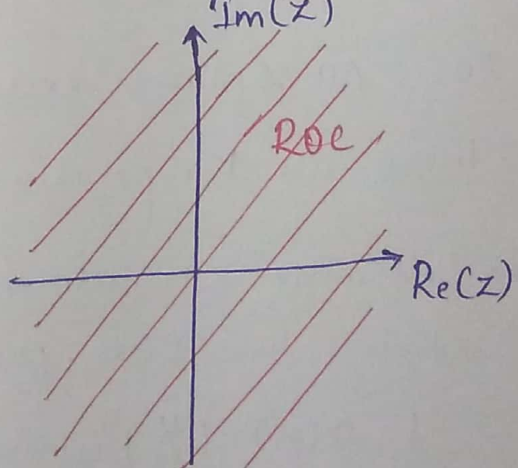
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$\delta(n)$ exist only when $n = 0$

$X(z) = 1 \cdot z^0 = 1$.

ROC = entire z plane.

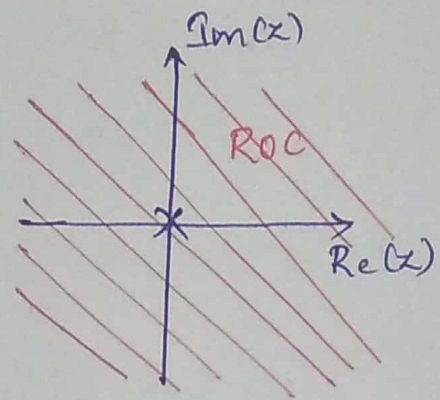
No zero's no poles



2. $x(n) = \delta(n-K)$; $K > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-K) z^{-n}$$



$\delta(n-k)$ exist only when $n=K$.

$\therefore$ $X(z) = 1 . z^{-K} = 1/z^K$

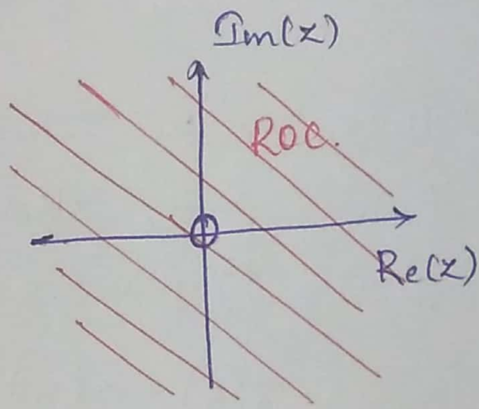
ROC : All z plane except $z=0$

$\therefore$ there are K repetative poles at origin.

3. $x(n) = \delta(n+K)$; $K > 0$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n+K) z^{-n}$$



$\delta(n+k)$ exisit only when $n=-K$.

$\therefore$ $X(z) = 1 . z^K = z^K$.

ROC : All z plane exupt $z=0$.

there are K repetative zeroes at origin.

7. Determine z transform and ROC for the given signals. Sketch the ROC, poles and zeroes.

1. $x(n) = (1/2)^n u(n-2)$

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

$$x(z) = \sum_{n=-\infty}^{\infty} [(1/2)^n u(n-2)] z^{-n}$$

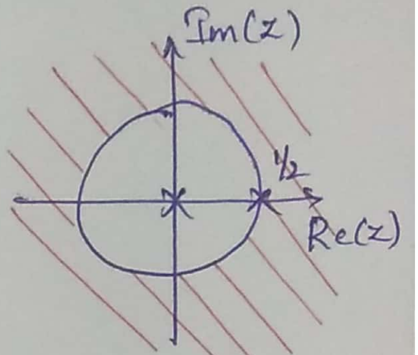
$$= \sum_{n=2}^{\infty} (1/2)^n z^{-n}$$

$$= \sum_{n=2}^{\infty} (1/2 z^{-1})^n$$

$$\boxed{\sum_{n=2}^{\infty} \alpha^n = \frac{\alpha^2}{1-\alpha}}$$

$$x(z) = \frac{(1/2 z^{-1})^2}{1 - 1/2 z^{-1}} = \frac{1}{4z^2(1-1/2 z^{-1})}$$

$$= \frac{1}{4z(z-1/2)}$$



ROC : $|1/2 z^{-1}| < 1 \Rightarrow |1/2| < |z|$

2. $x(n) = 2^n u(-n-1)$

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

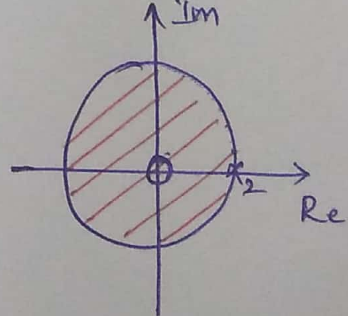
$$= \sum_{n=-\infty}^{\infty} 2^n u(-n-1) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} 2^n \cdot z^{-n}$$

$$= \sum_{n=1}^{\infty} 2^{-n} \cdot z^{n} = \sum_{n=1}^{\infty} (2^{-1} z)^n$$

$$\boxed{\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha}}$$

$$= \frac{2^{-1} z}{1 - 2^{-1} z} = \frac{-z}{z-2}$$



ROC : $|z| < 2$. hole at $z = 0$.

Pole at $z = 2$.