3. $x(n) = (1/2)^{|n|}$

this is a two-sided infinite Sequence.

By defination of z transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n).z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} (1/2)^{|n|}.z^{-n}.$$

$$= \sum_{n=-\infty}^{-1} (1/2)^{-n} z^{-n} + \sum_{n=0}^{\infty} (1/2)^{n} z^{-n}.$$

$$= \sum_{n=1}^{\infty} 2^{-n} z^{+n} + \sum_{n=0}^{\infty} (1/2\, z^{-1})^{n}.$$

$$= \sum_{n=1}^{\infty} (2^{-1} z)^{n} + \sum_{n=0}^{\infty} (1/2\, z^{-1})^{n}.$$

$$= \frac{2^{-1} z}{1-2^{-1} z} + \frac{1}{1-1/2\, z^{-1}}. \quad -(1)$$

$$= \frac{-z}{(z-2)} + \frac{z}{(z-1/2)}$$

$$= \frac{-3/2\, z}{(z-2)(z-1/2)}$$

from eq (1) there are two series.

1st Series converge if $|2^{-1} z| < 1 \Rightarrow |z/2| < 1$

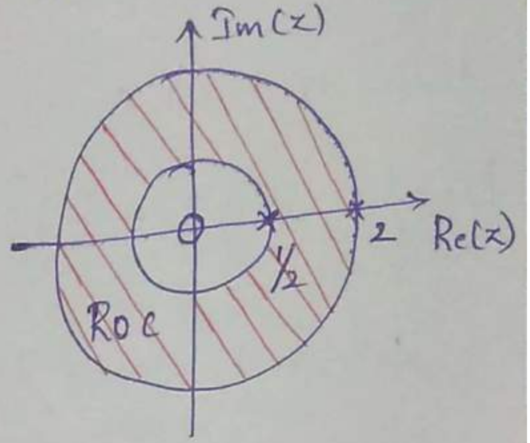
$|z| < |2|$

Scanned with CamScanner

the Second Series Converges if $|\frac{1}{2}z^{-1}| < 1 \Rightarrow |\frac{1}{2z}| < 1$.

$$\therefore \ |1/2| < |z|$$

$$\Rightarrow \text{ROC}: \ |1/2| < |z| < |2|$$



* Properties of Z transforms :-

1. Linearity :-

If $x_1(n) \xleftrightarrow{ZT} X_1(z)$ and $x_2(n) \xleftrightarrow{FT} X_2(z)$

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{ZT} a_1 X_1(z) + a_2 X_2(z)$ and.

ROC is the intersection of ROC of $X_1(z)$ and $X_2(z)$.

Proof :- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

$$= \sum_{n=-\infty}^{\infty} [a_1 x_1(n) + a_2 x_2(n)] z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}.$$

$$= a_1 \sum_{n=-\infty}^{\infty} x_1(n) z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n) z^{-n}.$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

Hence proved.

2. Time Shifting or Translation :-

Scanned with CamScanner

If $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$

then $x(n-k) \xleftrightarrow{ZT} z^{-k} X(z)$ ROC: $r_1 < |z| < r_2$.

Proof: $$Z\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-k)\, z^{-n}.$$

Let $n-k = m$, then $n = m+k$. & $m = -\infty$ to ∞

$$Z\{x(n-k)\} = \sum_{m=-\infty}^{\infty} x(m)\, z^{-(k+m)}$$

$$Z\{x(n-k) = \sum_{m=-\infty}^{\infty} x(m)\, z^{-m} \cdot z^{-k}.$$

$$= z^{-k} \sum_{m=-\infty}^{\infty} x(m)\, z^{-m}.$$

$$= z^{-k} X(z)$$

hence proved.

3. Scaling in z-Domain or Multiplication by Exponent

Let $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$ then.

$$a^n x(n) \xleftrightarrow{ZT} X(z/a) \quad \text{ROC}: |a| r_1 < |z| < |a| r_2$$

Proof :- $$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot a^n z^{-n}$$

$$= \sum^{\infty} x(n)\, (a^{-1} z^{1})^{-n}$$

$= X(a^{-1}z)$

$= X(z/a)$

ROC: $r_1 < |z/a| < r_2$

$\Rightarrow |a| r_1 < |z| < r_2 |a|$

Hence proved.

4. Time Reversal:

Let $x(n) \xleftrightarrow{z} X(z)$ ROC: $r_1 < |z| < r_2$ then

$x(-n) \xleftrightarrow{z} X(z^{-1})$ ROC: $\frac{1}{r_2} < |z| < \frac{1}{r_1}$

Proof: $Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$

Replace $-n = m \Rightarrow n = -m$.

$Z\{x(-n)\} = \sum_{m=\infty}^{-\infty} x(m) z^{m}$

$= \sum_{m=-\infty}^{\infty} x(m) (z^{-1})^{-m}$

$= X(+z^{-1})$

$\therefore$ ROC; $r_1 < |z^{-1}| < r_2$

$\Rightarrow \frac{1}{r_2} < |z| < \frac{1}{r_1}$

Scanned with CamScanner

5. Differentiation in Z-Domain or Multiplication by a Ramp

Let $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$ then

$$n\,x(n) \xleftrightarrow{ZT} -z\frac{d}{dz}X(z) \quad \text{ROC: } r_1 < |z| < r_2.$$

Proof :- $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$.

differenting both the sides with respect to z.

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} \frac{d}{dz}\left[x(n). z^{-n}\right]$$

$$\frac{d}{dz}X(z) = \sum_{n=-\infty}^{\infty} x(n)(-n) z^{-n-1}$$

$$\frac{d}{dz}X(z) = -\sum_{n=-\infty}^{\infty} n.x(n) z^{-n}. z^{-1}$$

$$\frac{d}{dz}X(z) = -z^{-1}\sum_{n=-\infty}^{\infty} n x(n). z^{-1}$$

$$\frac{d}{dz}X(z) = -z^{-1}. Z\{n x(n)\}.$$

$$-z\frac{d}{dz}X(z) = Z\{n x(n)\}$$

hence proved.

6. Convolution in Time Domain:-

Let $x_1(n) \xleftrightarrow{ZT} X_1(z)$ ROC: $a_1 < |z| < b_1$

and $x_2(n) \xleftrightarrow{ZT} X_2(z)$ Roc: $a_2 < |z| < b_2$ then

Scanned with CamScanner

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) \cdot X_2(z)$$

the ROC is the intersection of ROC's of $X_1(z)$ and $X_2(z)$.

Proof: By defination of convolution

$$x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

$$\therefore Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k)\right] z^{-n}$$

Interchanging the order of summation

$$= \sum_{k=-\infty}^{\infty} x_1(k) \left\{\sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}\right\}$$

we know that $x_2(n-k) \xleftrightarrow{ZT} z^{-k} X_2(z)$

[using time shifting property]

$$Z\{x_1(n) * x_2(n)\} = \sum_{k=-\infty}^{\infty} x_1(k) \left\{z^{-k} X_2(z)\right\}$$

$$= \left\{\sum_{k=-\infty}^{\infty} x_1(k) \cdot z^{-k}\right\} X_2(z)$$

$$Z\{x_1(n) * x_2(n)\} = X_1(z) \cdot X_2(z)$$

7. <u>Conjugation of a Complex Sequence</u>:-

Let $x(n) \xleftrightarrow{ZT} X(z)$ ROC: $r_1 < |z| < r_2$

$$x^*(n) \xleftrightarrow{ZT} X^*(z^*)$$

Scanned with CamScanner

Proof: $Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$

$$= \sum_{n=-\infty}^{\infty} [x(n) (z^*)^{-n}]^*$$

$$= \left[\sum_{n=-\infty}^{\infty} x(n) (z^*)^{-n}\right]^*$$

$$= [X(z^*)]^*$$

$$= X^*(z^*)$$

8. Initial Value theorem:-

If $x(n) = 0$ for $n < 0$ (i.e $x(n)$ is casual) then

$$\lim_{n\to 0} x(n) = x(0) = \lim_{z\to\infty} X(z)$$

Proof:- $Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}$

$$= x(0) + x(1) z^{-1} + x(2) z^{-2} + \ldots$$

Taking Limits $z \to \infty$ on both the sides.

$$\lim_{z\to\infty} X(z) = x(0) + 0 + 0$$

$$\lim_{z\to\infty} X(z) = x(0) = \lim_{n\to 0} x(n)$$

hence proved.

Scanned with CamScanner

9. Final Value theorem:-

If $x(n) \xleftrightarrow{ZT} X(Z)$ and if $X(Z)$ exists with no poles outside the unit Circle and it has no double or higher order poles on unit circle Centered at the origin of the Z plane, then

$$\lim_{n \to \infty} x(n) = x(\infty) = \lim_{Z \to 1} (Z-1) X(Z)$$

Proof:- $Z\{x(n)\} = X(Z)$ — ①

$Z\{x(n+1)\} = Z X(Z)$ — ② [Time shifting]

Subtract eq ① from eq ②

$$Z\{x(n+1)\} - Z\{x(n)\} = Z X(Z) - X(Z)$$

$$\sum_{n=0}^{\infty} x(n+1) Z^{-n} - \sum_{n=0}^{\infty} x(n) Z^{-n} = (Z-1) X(Z)$$

$$\sum_{n=0}^{\infty} \{x(n+1) - x(n)\} Z^{-n} = Z\{x(n+1) - x(n)\}$$

$$Z\{x(n+1) - x(n)\} = \sum_{n=0}^{\infty} \{x(n+1) - x(n)\} Z^{-n}$$

taking Limit as $n \to \infty$.

$$= \lim_{n \to \infty} \left[\sum_{K=0}^{\infty} \{x(K+1) - x(K)\} Z^{-K} \right]$$

Scanned with CamScanner

$$= \lim_{n\to\infty}\Big[x(1)z^0 + x(2)z^{-1} + \dots + x(n+1)z^{-n} - x(0) - x(1)z^{-1} - x(2)z^{-2} \dots - x(n)z^{-n}\Big]$$

$$= \lim_{n\to\infty}\Big[-x(0) + x(1)(1-z^{-1}) + x(2)(z^{-1}-z^{-2}) + \dots + x(n+1)z^{-n}\Big]$$

taking limits as $z \to 1$ on both sides, we get

$$\lim_{z\to 1} Z\{x(n+1) - x(n)\} = \lim_{n\to\infty}\Big[-x(0) + x(n+1)\Big]$$

$$\lim_{z\to 1}\Big[zX(z) - X(z) - zx(0)\Big] = -x(0) + \lim_{n\to\infty} x(n)$$

$$\lim_{z\to 1}\Big[(z-1)X(z) - zx(0)\Big] = -x(0) + \lim_{n\to\infty} x(n)$$

$$\lim_{z\to 1}\Big[(z-1)X(z)\Big] - \cancel{x(0)} = -\cancel{x(0)} + \lim_{n\to\infty} x(n)$$

$$\boxed{\lim_{n\to\infty} x(n) = \lim_{z\to 1} (z-1)X(z)}$$

hence proved.

Eg 1. Determine the z-transform of the following.

a. $x_1(n) = \delta(n-k)$

$\delta(n) \xleftrightarrow{ZT} 1$ ROC: entire z plane.

using time shift property

i.e $x(n-k) \xleftrightarrow{ZT} z^{-k}X(z)$

Scanned with CamScanner

$$z[\delta(n-k)] = \sum_{n=-\infty}^{\infty} \delta(n-k)\, z^{-(n-k)} = z^{-k}\, z\{\delta(n)\}$$

$$= \delta(0)\, z^{0} \cdot z^{-k}$$

$$= 1 \cdot z^{-k}$$

b. $x_2(n) = \delta(n+k)$

$$z[\delta(n+k)] = z^{k} \cdot z\{\delta(n)\}$$ using time shift

$$= z^{k} \cdot 1$$

$$= z^{k}$$

c. $x_3(n) = u(-n)$

By using time reversal property.

$$x(-n) \xleftrightarrow{ZT} X(z^{-1}) \quad \text{ROC}: \frac{1}{r_2} < |z| < \frac{1}{r_1}$$

$$z\{u(n)\} = \sum_{n=-\infty}^{\infty} x(n) \cdot z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} u(n) \cdot z^{-n} = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} = \frac{1}{1-z^{-1}}$$

$$z\{u(-n)\} = \sum_{n=-\infty}^{0} u(-n)\, z^{n} = \sum_{n=0}^{-\infty} 1 \cdot z^{+n}$$

$$= \frac{1}{1-z^{1}} = \frac{1}{1-z}$$

Scanned with CamScanner

d. $x_4(n) = n \cdot a^n u(n)$

$$z\{a^n u(n)\} = \frac{1}{1-az^{-1}}$$

we know that $z\{n \cdot x(n)\} = -z \frac{d}{dz} X(z)$, differentiation in z-domain property

$$z\{n \cdot \underbrace{a^n u(n)}_{x(n)}\} = -z \frac{d}{dz}\left[\frac{1}{1-az^{-1}}\right]$$

$$= -z\left[\frac{(1-az^{-1})\, d/dz(1) - 1 \cdot d/dz\,(1-az^{-1})}{(1-az^{-1})^2}\right]$$

$$= -z\left[\frac{(1-az^{-1})\,0 - 1 \cdot a(-1)z^{-2}}{(1-az^{-1})^2}\right]$$

$$= \frac{-z \times az^{-2}}{(1-az^{-1})^2}$$

$$= \frac{az^{-1}}{(1-az^{-1})^2} \quad ; \text{ROC} : |z| > |a|$$

2. Determine the z transform of $x(n) = \begin{cases} a^n & \text{for } 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$

By defination of z transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Scanned with CamScanner

$$x(z) = \sum_{n=0}^{N-1} a^n z^{-n} = \sum_{n=0}^{N-1} (az^{-1})^n$$

$$x(z) = \frac{(az^{-1})^0 - (az^{-1})^{N-1+1}}{1 - az^{-1}}$$

$$\boxed{\sum_{n=N_1}^{N_2} a^n = \frac{a^{N_1} - a^{N_2+1}}{1-a} ; N_2 > N_1}$$

$$x(z) = \frac{1 - (az^{-1})^N}{1 - az^{-1}}$$

3. Determine z transform of $x(n) = \cos \Omega_0 n \cdot u(n)$

$$x(n) = \cos \Omega_0 n \cdot u(n)$$

$$= \frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2} \cdot u(n)$$

$$= \frac{1}{2} e^{j\Omega_0 n} \cdot u(n) + \frac{1}{2} e^{-j\Omega_0 n} \cdot u(n)$$

$$x(z) = z\left\{\frac{1}{2} e^{j\Omega_0 n} \cdot u(n)\right\} + z\left\{\frac{1}{2} e^{-j\Omega_0 n} \cdot u(n)\right\}$$

this is of the form $a^n u(n) \xleftrightarrow{ZT} \frac{1}{1-az^{-1}}$

$$\therefore x(z) = \frac{1}{2}\left[\frac{1}{1 - e^{j\Omega_0} z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1 - e^{-j\Omega_0} z^{-1}}\right]$$

ROC: $|z| > |e^{j\Omega_0}|$ and $|z| > |e^{-j\Omega_0}|$

Here $e^{j\Omega_0} = \cos\Omega_0 + j \sin\Omega_0 \Rightarrow |e^{j\Omega_0}| = \sqrt{\cos^2\Omega_0 + \sin^2\Omega_0} = 1$

$e^{-j\Omega_0} = \cos\Omega_0 - j \sin\Omega_0 \Rightarrow |e^{-j\Omega_0}| = \sqrt{\cos^2\Omega_0 + \sin^2\Omega_0} = 1$

$\therefore$ ROC: $|z| > |1|$

Scanned with CamScanner

$$X(z) = \frac{1}{2}\left\{\frac{1}{1-e^{j\Omega_0}z^{-1}} + \frac{1}{1-e^{-j\Omega_0}z^{-1}}\right\}$$

$$= \frac{1}{2}\left\{\frac{1-e^{-j\Omega_0}z^{-1} + 1-e^{j\Omega_0}z^{-1}}{(1-e^{j\Omega_0}z^{-1})\,(1-e^{-j\Omega_0}z^{-1})}\right\}$$

$$= \frac{1}{2}\left\{\frac{2 - z^{-1}(e^{j\Omega_0} + e^{-j\Omega_0})}{1-z^{-1}(e^{j\Omega_0} + e^{-j\Omega_0}) + z^{-2}}\right\}$$

$$= \frac{1}{2}\left\{\frac{2 - z^{-1}\,2\cos\Omega_0}{1-z^{-1}.\,2\cos\Omega_0 + z^{-2}}\right\}$$

$$= \frac{1 - z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0 + z^{-2}} \qquad \text{ROC} : |z| > 1.$$

4. find the z transform of the signal

$$x(n) = (1/2)^n u(n) * (1/3)^n u(n)$$

Using Convolution property in time domain.

$$x_1(n) * x_2(n) \xleftrightarrow{ZT} X_1(z) . X_2(z)$$

$$x(n) = x_1(n) * x_2(n)$$

$$x_1(n) = (1/2)^n u(n) \qquad x_2(n) = (1/3)^n u(n)$$

this is of the form $a^n u(n)$

$$X_2(z) = \frac{1}{1-(1/3)z^{-1}}$$

$$X_1(z) = \frac{1}{1-(1/2)z^{-1}}$$

$$\therefore \; x_1(n) * x_2(n) \longleftrightarrow X_1(z) . X_2(z)$$

$$(1/2)^n u(n) * (1/3)^n u(n) \longleftrightarrow \frac{1}{1-1/2\,z^{-1}} \cdot \frac{1}{(1-1/2\,z^{-1})}$$

Scanned with CamScanner

5. Given $x(n) \xleftrightarrow{z} X(z) = \frac{z}{z^2+4}$ with ROC: $|z| < 2$

Using Z transform properties, determine z-transform of

1. $y(n) = n x(n)$

Using differentiation in Z domain property.

$$y(n) = n x(n) \xleftrightarrow{ZT} Y(z) = -z \frac{d}{dx} X(z)$$

$$\therefore Y(z) = -z \frac{d}{dz}\left[\frac{z}{z^2+4}\right]$$

$$= \frac{z^3 - 4z}{(z^2+4)^2}.$$

2. $y(n) = 2^n x(n)$

Using Scaling in z-domain property.

$$y(n) = 2^n x(n) \xleftrightarrow{ZT} Y(z) = X(z/2)$$

$$X(z) = \frac{z}{z^2+4} = \frac{z/2}{(z/2)^2+4}.$$

6. Find the initial and final value of the signal

$$X(z) = \frac{(z-3)z}{(z-1)(z-0.4)}$$

Initial value theorem: $x(0) = \lim_{z \to \infty} X(z)$

$$= \lim_{z \to \infty}\left[\frac{(z-3)z.}{(z-1)(z-0.4)}\right]$$

Scanned with CamScanner

$$= \lim_{z\to\infty} \frac{(1-3/z)}{(1-1/z)(1-\frac{0.4}{z})}$$

$$= 1.$$

$\therefore$ $x(0)$ = initial value = 1.

final value , $x(\infty) = \lim_{z\to 1} (1-z^{-1})\, X(z)$

$$= \lim_{z\to 1}\left[(1-z^{-1}) \cdot \frac{(z-3)z}{(z-1)(z-0.4)}\right]$$

$$= \lim_{z\to 1}\left[\frac{\cancel{(z-1)}}{\cancel{z}} \cdot \frac{(z-3).\cancel{z}}{\cancel{(z-1)}(z-0.4)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z-3}{z-0.4}\right]$$

$x(\infty) = -\frac{10}{3}$ = final Value.

7. find the value of $x(0)$ for the Sequence that has Z transform.

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \; ; \; ROC : |z| > 1/2.$$

$$X(z) = \frac{z}{z-1/2}$$

$$x(0) = \lim_{z\to\infty} X(z)$$

$$= 1.$$

Scanned with CamScanner

* Inverse Z - transforms :-

- Here we will recover the time-domain signal $x(n)$ from its Z transform $X(z)$.
- there are two possible approaches for finding the inverse Z-transform.

1. Partial fraction Expansion method.
2. Power Series Expansion method.

* Partial fraction expansion method :-

- Following steps are performed

Step 1 : Arrange the given $X(z)$ as.

$$\frac{X(z)}{z} = \frac{\text{Numerator polynomial}}{(z-p_1)(z-p_2)\cdots(z-p_n)}$$

Step 2 :

$$\frac{X(z)}{z} = \frac{A_1}{(z-p_1)} + \frac{A_2}{(z-p_2)} + \frac{A_3}{(z-p_3)} + \cdots + \frac{A_n}{(z-p_n)} \quad -(1)$$

where $A_K = (z-p_K) \cdot \left.\frac{X(z)}{z}\right|_{z=p_K}$ $\quad K = 1, 2, 3 \ldots N$.

Step 3 : Equation (1) can be written as.

$$X(z) = \frac{A_1 z}{(z-p_1)} + \frac{A_2 z}{(z-p_2)} + \cdots + \frac{A_n z}{(z-p_n)}$$

$$= \frac{A_1}{(1-p_1 z^{-1})} + \frac{A_2}{(1-p_2 z^{-1})} + \cdots + \frac{A_n}{(1-p_n z^{-1})}$$

Step 4 : All the terms in above steps are of the form

$\frac{A_K}{1-P_K z^{-1}}$. Depending upon ROC, following standard z-transform pairs must be used.

$$P_K^n u(n) \xleftrightarrow{z} \frac{1}{1-P_K z^{-1}}, \quad ROC: |z| > |P_k| \text{ (casual sequence)}$$

$$-(P_K)^n u(-n-1) \xleftrightarrow{z} \frac{1}{1-P_K z^{-1}} \quad ROC: |z| < |P_k| \text{ (non Casual sequence).}$$

Eg 1. Determine inverse z-transform of $x(z)$ given.

$$x(z) = \frac{1}{1-1.5z^{-1}+0.5z^{-2}} \text{ for the following ROC}$$

a. $|z| > 1$ b. $|z| < 0.5$ c. $0.5 < |z| < 1$

Sol: Step 1: First Convert $x(z)$ to positive powers of z i.e. x^{le} & ÷ through out by z^2.

$$x(z) = \frac{z^2}{z^2-1.5z+0.5}$$

$$\frac{x(z)}{z} = \frac{z}{z^2-1.5z+0.5} = \frac{z}{(z-1)(z-0.5)}$$

Step 2 = $$\frac{x(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-0.5)}$$

$$\therefore A_1 = (z-1) \cdot \frac{x(z)}{z}$$

$$A_2 = (z-0.5) \cdot \frac{x(z)}{z}$$

$$\therefore A_1 = (z-1) \frac{z}{(z-1)(z-0.5)} \bigg|_{z=1} = \frac{1}{1-0.5} = 2$$

Scanned with CamScanner

$$A_2 = (z-0.5) \times \left.\frac{z}{(z-1)(z-0.5)}\right|_{z=0.5} = \frac{0.5}{(0.5-1)} = -1$$

$$\therefore \frac{X(z)}{z} = \frac{2}{(z-1)} - \frac{1}{(z-0.5)}$$

$$X(z) = \frac{2z}{(z-1)} - \frac{z}{(z-0.5)}$$

$$= \frac{2}{(1-z^{-1})} - \frac{1}{(1-0.5z^{-1})}$$

Step 4: i. $x(n)$ for ROC of $|z| > 1$

- from above equation there are two poles.

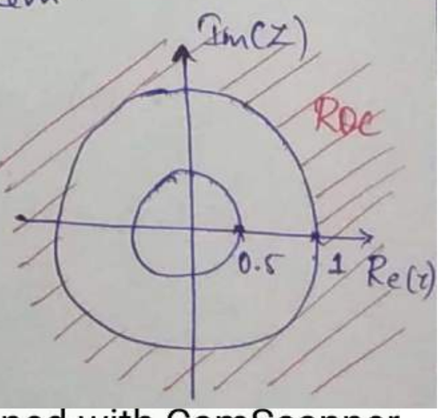
  i.e $z = 1$ and $z = 0.5$.

$\therefore$ given ROC condition indicates that the term $\frac{2}{(1-z^{-1})}$ must be a casual system / positive time exponential system.

- Since $|z| > 1$ by default $|z| > 0.5$

  cos $|z| > 1$ includes $|z| > 0.5$

  $\therefore$ the term $\frac{1}{(1-0.5z^{-1})}$ must be acausal system / Positive time exponential system.



$$\therefore x(n) = a^n u(n) - b^n u(n)$$

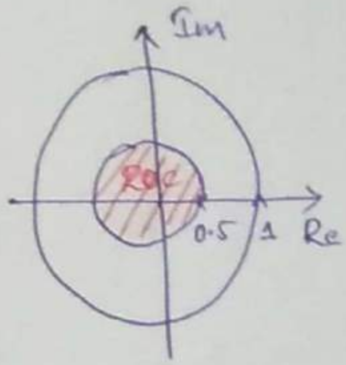
$$= 2(1)^n u(n) - 1\,(0.5)^n u(n)$$

$$= [2-(0.5)^n]\,u(n)$$

Scanned with CamScanner

b. $x(n)$ for ROC : $|z| < 0.5$. [$|z| < r$ = left / negative time sequence] 38

$\therefore$ it is a non casual system.



the sequence corrosponding to

$\frac{1}{(1-0.5z^{-1})}$ must be a non casual system

as indicated by ROC condition.

- if $|z| < 0.5$ by default to $|z| < 1$ there by

$\frac{2}{(1-z^{-1})}$ must also be a non casual system.

$$\Rightarrow x(n) = -b^n u(-n-1)$$

$$\therefore x(n) = 2[-1^n u(-n-1)] - 1[-(0.5)^n u(-n-1)]$$

$$= [-2 + (0.5)^n] u(-n-1)$$

c. $x(n)$ for ROC : $0.5 < |z| < |1|$

$\therefore$ ROC can be written as $|z| > 0.5$ and $|z| < 1$.

- there by the sequence $\frac{1}{(1-0.5z^{-1})}$ will be casual / positive time exponential sequence.

- the sequence $\frac{2}{(1-z^{-1})}$ will be non casual / negative time exponential sequence.

$$\therefore x(n) = -a^n u(-n-1) + b^n u(n)$$

$$x(n) = 2[-1^n u(-n-1)] - [(0.5)^n u(n)]$$

$$= -2u(-n-1) - (0.5)^n u(n)$$

Scanned with CamScanner

2. Find the inverse z-transform of x(z) using partial fraction expansion approach.

$$x(z) = \frac{z+1}{3z^2-4z+1} \quad ; \text{ROC } |z|>1.$$

$$\frac{x(z)}{z} = \frac{(z+1)}{z(3z^2-4z+1)} = \frac{(z+1)}{z(3z-1)(z-1)}$$

$$\frac{x(z)}{z} = \frac{(z+1)}{3z(z-1/3)(z-1)}$$

$$= \frac{A_1}{z} + \frac{A_2}{(z-1/3)} + \frac{A_3}{(z-1)}$$

$$A_1 = z \cdot \frac{x(z)}{z} = \cancel{z} \cdot \frac{(z+1)}{3\cancel{z}(z-1/3)(z-1)}\Bigg|_{z=0} = 1.$$

$$A_2 = \cancel{(z-1/3)} \cdot \frac{(z+1)}{3z\cancel{(z-1/3)}(z-1)}\Bigg|_{z=1/3} = -2.$$

$$A_3 = \cancel{(z-1)} \cdot \frac{(z+1)}{3z(z-1/3)\cancel{(z-1)}}\Bigg|_{z=1} = 1.$$

$$\therefore \frac{x(z)}{z} = \frac{1}{z} + \frac{(-2)}{(z-1/3)} + \frac{1}{(z-1)}$$

$$x(z) = 1 - \frac{2z}{(z-1/3)} + \frac{z}{(z-1)}$$

$\therefore$ poles at $z = 1/3$ and $z = 1$.

Scanned with CamScanner

$x(n)$ for ROC : $|z|>1$.

the Sequence corrosponding to $\frac{z}{z-1}$ must be a Casual System / Sequence.

if $|z|>1$ by default $|z|>1/3$

$$\therefore\ x(n) = \delta(n) - 2(1/3)^n u(n) + (1)^n u(n)$$

$$x(n) = \delta(n) - 2(1/3)^n u(n) + u(n)$$

3. Determine inverse z-transform of

$$x(z) = \frac{1}{(1+z^{-1})(1-z^{-1})^2}, \quad \text{ROC} : |z|>1.$$

Convert $x(z)$ to positive power of z.

$$x(z) = \frac{z^3}{(z+1)(z-1)^2}$$

$$\frac{x(z)}{z} = \frac{z^{\cancel{3}\,2}}{(z+1)(z-1)^2 \cdot \cancel{z}} = \frac{z^2}{(z+1)(z-1)^2}$$

here there are multiple poles at $z=1$. therefore the partial fraction expansion will be.

$$\frac{x(z)}{z} = \frac{A_1}{(z+1)} + \frac{A_2}{(z-1)} + \frac{A_3}{(z-1)^2}$$

$$A_1 = (z+1)\frac{x(z)}{z} = \cancel{(z+1)} \cdot \frac{z^2}{\cancel{(z+1)}(z-1)^2}$$

$$= \left.\frac{z^2}{(z-1)^2}\right|_{z=-1} = 1/4$$

$$\boxed{A_1 = 1/4}$$

Scanned with CamScanner

~~If~~ if the pole of multiplicity is found in $\frac{X(z)}{z}$ i.e

$$\frac{X(z)}{z} = \frac{\text{Numerator polynomial}}{(z-p)^n}$$

$$= \frac{A_1}{(z-p)} + \frac{A_2}{(z-p)^2} + \cdots + \frac{A_n}{(z-p)^n}$$

where A_1, A_2, A_n are given as.

$$A_K = \frac{1}{(n-K)!} \times \frac{d^{n-K}}{dz^{n-K}} \left\{ (z-p)^n \cdot \frac{X(z)}{z} \right\}_{z=p}$$

$\therefore$ in the given problem $A_K = A_2 \therefore K=2$ and $n=2$

$$\Rightarrow A_2 = \frac{1}{(2-2)!} \cdot \frac{d^{(2-2)}}{dz^{(2-2)}} \left\{ (z-1)^2 \cdot \frac{X(z)}{z} \right\}$$

$$= \frac{1}{0!} \cdot \frac{d^0}{dz^0} \left\{ \cancel{(z-1)^2} \cdot \frac{z^2}{(z+1)\cancel{(z-1)^2}} \right\}$$

$$= 1 \cdot \frac{d}{dz} \left\{ \frac{z^2}{(z+1)} \right\}$$

$$= \left. \frac{(z+1)2z - z^2}{(z+1)^2} \right|_{z=1} = 3/4$$

$$\boxed{A_2 = 3/4}$$

$$A_3 = \left. (z-1)^2 \cdot \frac{X(z)}{z} \right|_{z=1}$$

$$= \left. \frac{z^2}{(z+1)} \right|_{z=1} = \frac{1}{2}$$

$$\boxed{A_3 = 1/2}$$

Scanned with CamScanner

$$\frac{X(z)}{z} = \frac{1/4}{(z+1)} + \frac{3/4}{(z-1)} + \frac{1/2}{(z-1)^2}$$

$$X(z) = \frac{1/4 \cdot z}{(z+1)} + \frac{3/4\, z}{(z-1)} + \frac{1/2\, z}{(z-1)^2}$$

$$X(z) = \frac{1/4}{(1+z^{-1})} + \frac{3/4}{(1-z^{-1})} + \frac{1/2\, z^{-1}}{(1-z^{-1})^2}$$

given ROC : $|z| > 1$.

$\therefore$ the Sequence corrosponding to $\frac{1/4}{(1+z^{-1})}$, $\frac{3/4}{(1-z^{-1})}$, $\frac{1/2\, z^{-1}}{(1-z^{-1})^2}$ must be a Casual System/Sequence as $|z| > 1$ for all the three sequence.

$\therefore x(n) = a^n u(n)$ for first two terms.

$\Rightarrow x(n) = 1/4\, (-1)^n u(n) + 3/4\, (1)^n u(n)$

for third term of $X(z)$ it is of the form $\frac{az^{-1}}{(1-az^{-1})^2}$

$\therefore n a^n u(n) \xleftrightarrow{ZT} \frac{az^{-1}}{(1-az^{-1})^2}$ [page 31 problem 1d refere]

$\therefore x(n) = 1/4\, (-1)^n u(n) + 3/4\, (1)^n u(n) + \frac{1}{2} n (1)^n u(n)$

$= \left[1/4 (-1)^n + 3/4 (1)^n + \frac{1}{2} n \right] u(n)$

Scanned with CamScanner

4. Determine the sequence whose z-transform is $X(z) = \dfrac{1+2z^{-1}+z^{-2}}{1-\frac{3}{2}z^{-1}+\frac{1}{2}z^{-2}}$, ROC: $|z|>1$.

In the above given problem the highest power of numerator and denominator in $X(z)$ are same. Such expression cannot be expanded in partial fraction therefore we need to perform division.

$$\Rightarrow X(z) = \frac{z^{-2}+2z^{-1}+1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1}$$

$$\begin{array}{r|l} & \quad 2 \\ \hline \frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1 & z^{-2}+2z^{-1}+1 \\ & \underset{(-)}{z^{-2}} \underset{(+)}{-3z^{-1}} \underset{(-)}{+2} \\ \hline & \quad 5z^{-1}-1 \end{array}$$

$$X(z) = 2 + \frac{5z^{-1}-1}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1} = 2 + \frac{-1+5z^{-1}}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1}$$

$$= 2 + X_1(z)$$

$$X_1(z) = \frac{-1+5z^{-1}}{\frac{1}{2}z^{-2}-\frac{3}{2}z^{-1}+1} \times \frac{z^2}{z^2}$$

Converting to positive power of z we have.

$$X_1(z) = \frac{(-z+5)z}{z^2-\frac{3}{2}z+\frac{1}{2}z}.$$

$$\frac{X_1(z)}{z} = \frac{(+5-z)}{z^2-\frac{3}{2}z+\frac{1}{2}z}$$

Scanned with CamScanner

$$\frac{X_1(z)}{z} = \frac{-z+5}{(z-1)(z-1/2)}$$

$$\Rightarrow \frac{X_1(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z-1/2)}$$

$$A_1 = (z-1)\cdot\frac{X_1(z)}{z} = \frac{-z+5}{\cancel{(z-1)}(z-1/2)} \times \cancel{(z-1)}$$

$$= \left.\frac{-z+5}{(z-1/2)}\right|_{z=1} = 8. \qquad \boxed{A_1 = 8}$$

$$A_2 = (z-1/2)\cdot\frac{X_1(z)}{z} = \cancel{(z-1/2)}\cdot\frac{-z+5}{(z-1)\cancel{(z-1/2)}}$$

$$= \left.\frac{-z+5}{(z-1)}\right|_{z=1/2} = -9 \qquad \boxed{A_2 = -9}$$

$$\Rightarrow \frac{X_1(z)}{z} = \frac{8}{(z-1)} - \frac{9}{(z-1/2)}$$

$$X_1(z) = \frac{8z}{(z-1)} - \frac{9z}{(z-1/2)} = \frac{8}{(1-z^{-1})} - \frac{9}{(1-1/2\, z^{-1})}$$

$$\therefore X(z) = 2 + X_1(z)$$

$$= 2 + \frac{8}{(1-z^{-1})} - \frac{9}{(1-1/2\, z^{-1})}$$

given ROC : $|z| > 1$.

the ROC of the above expression $\frac{8}{1-z^{-1}}$ is $|z|>1$.

$\frac{9}{1-1/2\, z^{-1}}$ is $|z| > 1/2$ if $|z|>1$ then it includes $|z| > 1/2$ there by the sequence is casual.

Scanned with CamScanner

$$\therefore x(n) = 2\delta(n) + 8(1)^n u(n) - 9(1/2)^n u(n).$$ (45)

$$= 2\delta(n) + \left[8(1)^n - 9(1/2)^n\right] u(n).$$

(5) $X(z) = \log\left(\frac{1}{1-az^{-1}}\right)$; ROC $|z|>|a|$. find the inverse z-transform of the following expression given.

$$X(z) = \log\left(\frac{1}{1-az^{-1}}\right) = -\log(1-az^{-1})$$

$$\log(1-p) = -\sum_{n=1}^{\infty} \frac{p^n}{n} \qquad |p|<1.$$

$$\therefore X(z) = -\left[-\sum_{n=1}^{\infty} \frac{(az^{-1})^n}{n}\right]$$

$$= \sum_{n=1}^{\infty} \left(\frac{a^n}{n}\right) z^{-n}.$$

Comparing it with the standard defination of Z transform $x(n) = \begin{cases} \frac{a^n}{n} & \text{for } n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$

$$\Rightarrow x(n) = \frac{a^n}{n} u(n-1)$$

* Power Series expansion method. [long division]

- By defination of Z-transform of the Sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}.$$

Scanned with CamScanner

$$= \cdots + x(-2)z^{2} + x(-1)z + x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

from the above expansion of z transform, the sequence $x(n)$ can be obtained as.

$$x(n) = \{\cdots, x(-2), x(-1), x(0), x(1), x(2), \cdots\}$$

$\therefore$ the power series expansion can be obtained directly or by long division method.

1. Determine inverse z transform of the following.

a. $X(z) = \dfrac{1}{1-az^{-1}}$; ROC : $|z| > |a|$

$$
\begin{array}{r|l}
 & 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} \\
\hline
1-az^{-1} & \cancel{1} \\
 & \underline{\cancel{1} - az^{-1}} \\
 & a\cancel{z^{-1}} \\
 & \underline{a\cancel{z^{-1}} - a^2z^{-2}} \\
 & a^2\cancel{z^{-2}} \\
 & \underline{a^2\cancel{z^{-2}} - a^3z^{-3}} \\
 & a^3z^{-3} \\
 & \underline{a^3z^{-3} - a^4z^{-4}} \\
 & a^4z^{-4} \cdots
\end{array}
$$

$\therefore X(z) = 1 + az^{-1} + a^2z^{-2} + a^3z^{-3} + \cdots$

Taking inverse z transform.

$$x(n) = \{1, a, a^2, a^3, \cdots\}$$

② Using long division method or power series expansion technique, find the inverse z-transform of the following $X(z)$

a. $X(z) = \dfrac{z}{2z^2 - 3z + 1}$; ROC : $|z| < 1/2$

b. $X(z) = \dfrac{z}{2z^2 - 3z + 1}$; ROC : $|z| > 1$.

a. $X(z)$ has poles at $z = 1/2$ and $z = 1$.

- given ROC : $|z| < 1/2$ thereby the poles $z = 1/2$ & $z = 1$ are outside the circle of ROC. this implies $x(n)$ is a negative time sequence term.
- there by quotient must have only positive powers of z.

$$
\begin{array}{r|l}
 & z + 3z^2 + 7z^3 + 15z^4 + \dots \\
\hline
1 - 3z + 2z^2 & \cancel{z} \\
 & \cancel{z} - 3z^2 + 2z^3 \\
\hline
 & 3\cancel{z^2} - 2z^3 \\
 & 3\cancel{z^2} - 9z^3 + 6z^4 \\
\hline
 & 7z^3 - 6z^4 \\
 & 7z^3 - 21z^4 + 14z^5 \\
\hline
 & 15z^4 - 14z^5 \dots
\end{array}
$$

$\therefore X(z) = \dots\dots + 15z^4 + 7z^3 + 3z^2 + z$.

$$x(n) = \{\dots\dots 15, 7, 3, 1, \underset{\uparrow}{0}\}$$

Scanned with CamScanner

b. In this case poles at $z = 1/2$ and $z = 1$ are inside the Circle of Radius 1 in the z plane therefore $x(n)$ must be a right handed Sequence with quotient having only negative powers of z.

$$\frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{18}{16}z^{-4} + \cdots$$

$$2z^2 - 3z + 1 \;\big)\; \cancel{z}$$

$$\cancel{z} - \frac{3}{2} + \frac{1}{2}z^{-1}$$

$$\frac{3}{\cancel{2}} - \frac{1}{2}z^{-1}$$

$$\cancel{\frac{3}{2}} - \frac{9}{4}z^{-1} + \frac{3}{4}z^{-2}$$

$$\frac{7}{4}\cancel{z^{-1}} - \frac{3}{4}z^{-2}$$

$$\cancel{\frac{7}{4}z^{-1}} - \frac{21}{8}z^{-2} + \frac{7}{8}z^{-3}$$

$$\frac{18}{8}z^{-2} - \frac{7}{8}z^{-3}$$

$$\therefore\; X(z) = \frac{1}{2}z^{-1} + \frac{3}{4}z^{-2} + \frac{7}{8}z^{-3} + \frac{18}{16}z^{-4} + \cdots$$

Hence $x(n) = \{\underset{\uparrow}{0}, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{18}{16}, \cdots\}$

(3) Find the inverse z transform of

$$X(z) = z^2\left(1 - \frac{1}{2}z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right)$$

$$= \left(z^2 - \frac{1}{2}z\right)\left(1 - z^{-2}\right)$$

$$= z^2 - \frac{1}{2}z - 1 + \frac{1}{2}z^{-1}.$$

$$\therefore\; x(n) = \{1, -\frac{1}{2}, \underset{\uparrow}{-1}, \frac{1}{2}\}$$

$$= \delta(n+2) - \frac{1}{2}\delta(n+1) - \delta(n) + \frac{1}{2}\delta(n-1)$$

Scanned with CamScanner

* Transform Analysis of LTI System :-

Ztransform plays an important role in the analysis and representation of discrete-time LTI system.

1. Transfer function and impulse response :-

Consider an output of LTI system. We know that o/p of LTI is given by.

$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K) = x(n) * h(n) \quad -(1)$$

here $y(n)$ is the o/p of a LTI system.

$x(n)$ is the i/p to the system.

$h(n)$ is the impulse response of the system.

Let $x(n) \xleftrightarrow{Z} X(Z)$ & $y(n) \xleftrightarrow{Z} Y(Z)$

$h(n) \xleftrightarrow{Z} H(Z)$

Taking Z transform of equation (1) we get.

$$Y(Z) = Z\{x(n) * h(n)\}$$

By Convolution property

$$Y(Z) = X(Z) \cdot H(Z)$$

$$\boxed{H(Z) = \frac{Y(Z)}{X(Z)}}$$

$H(Z)$ is called the transfer function or System func"

* $h(n)$ is the impulse response of the System.

Scanned with CamScanner

1. A Casual System has i/p $x(n)$ and o/p $y(n)$. Find the impulse response of the system if.

$$x(n) = \delta(n) + \tfrac{1}{2}\delta(n-1) - \tfrac{1}{8}\delta(n-2)$$

$$y(n) = \delta(n) - \tfrac{3}{4}\delta(n-1)$$

Taking z transform of the equation given

$$X(z) = 1 + \tfrac{1}{2}z^{-1} - \tfrac{1}{8}z^{-2}$$

$$Y(z) = 1 - \tfrac{3}{4}z^{-1}$$

$\therefore$ Transfer function or system function

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 - \frac{3}{4}z^{-1}}{1 + \frac{1}{2}z^{-1} - \frac{1}{8}z^{-2}}$$

$$H(z) = \frac{1 - \frac{3}{4}z^{-1}}{(1 - \frac{1}{4}z^{-1})(1 + \frac{1}{2}z^{-1})}$$

Using partial fraction method to find impulse response

$$H(z) = \frac{z(z - \frac{3}{4})}{(z - \frac{1}{4})(z + \frac{1}{2})}$$ Convert to positive powers of z.

$$\frac{H(z)}{z} = \frac{(z - \frac{3}{4})}{(z - \frac{1}{4})(z + \frac{1}{2})}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{A_1}{(z - \frac{1}{4})} + \frac{A_2}{(z + \frac{1}{2})}$$

Scanned with CamScanner

$$A_1 = (z - 1/4) \cdot \frac{H(z)}{z} = \cancel{(z-1/4)} \cdot \frac{(z-3/4)}{\cancel{(z-1/4)}(z+1/2)}\Bigg|_{z=1/4}$$

$$A_1 = \frac{z-3/4}{z+1/2}\Bigg|_{z=1/4} = -2/3 \qquad \boxed{A_1 = -2/3}$$

$$A_2 = (z+1/2) \cdot \frac{H(z)}{z} = \cancel{(z+1/2)} \cdot \frac{(z-3/4)}{(z-1/4)\cancel{(z+1/2)}}\Bigg|_{z=-1/2}$$

$$A_2 = \frac{(z-3/4)}{(z-1/4)}\Bigg|_{z=-1/2} = 5/3 \qquad \boxed{A_2 = 5/3}$$

$$\Rightarrow \frac{H(z)}{z} = \frac{-2/3}{(z-1/4)} + \frac{5/3}{(z+1/2)}$$

$$H(z) = \frac{-2/3\,z}{(z-1/4)} + \frac{5/3\,z}{(z+1/2)}$$

$$H(z) = \frac{-2/3}{(1-1/4\,z^{-1})} + \frac{5/3}{(1+1/2\,z^{-1})}$$

It is given that the system is casual there fore $|z| > 1$.

$$\therefore h(n) = -2/3\,(1/4)^n u(n) + 5/3\,(-1/2)^n u(n)$$

$$= 1/3\left[-2(1/4)^n u(n) + 5(-1/2)^n u(n)\right]$$

2. A system has impulse response $h(n) = (1/2)^n u(n)$. Determine the input to the system if the o/p is given by

$$y(n) = 1/3\,u(n) + 2/3\,(-1/2)^n u(n)$$

Scanned with CamScanner

given: $y(n) = \frac{1}{3}u(n) + \frac{2}{3}(-\frac{1}{2})^n u(n)$

$$h(n) = (\tfrac{1}{2})^n u(n)$$

We have, $H(z) = \frac{Y(z)}{X(z)} \Rightarrow X(z) = \frac{Y(z)}{H(z)}$

$$Y(z) = \frac{1/3}{(1-z^{-1})} + \frac{2/3}{(1+\frac{1}{2}z^{-1})}$$

$$= \frac{\frac{1}{3}(1+\frac{1}{2}z^{-1}) + \frac{2}{3}(1-z^{-1})}{(1-z^{-1})(1+\frac{1}{2}z^{-1})}$$

$$= \frac{(1-\frac{1}{2}z^{-1})}{(1-z^{-1})(1+\frac{1}{2}z^{-1})}$$

$$H(z) = \frac{1}{(1-\frac{1}{2}z^{-1})}$$

$$\therefore X(z) = \frac{Y(z)}{H(z)} = \frac{(1-\frac{1}{2}z^{-1})^2}{(1-z^{-1})(1+\frac{1}{2}z^{-1})}$$

$$= \frac{1 - z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{1}{2}z^{-1} - \frac{1}{2}z^{-2}}$$

highest power of numerator = highest power of denominator

$$-\tfrac{1}{2}$$

$$-\tfrac{1}{2}z^{-2} - \tfrac{1}{2}z^{-1} + 1 \;\big)\; \tfrac{1}{4}z^{-2} - z^{-1} + 1$$

$$\tfrac{1}{4}z^{-2} + \tfrac{1}{4}z^{-1} - \tfrac{1}{2}$$

$$-\;\;\;\;\;\;-\;\;\;\;\;\;+$$

$$-\tfrac{5}{4}z^{-1} + \tfrac{3}{2}$$

$$\therefore X(z) = -\tfrac{1}{2} + \frac{-\tfrac{5}{4}z^{-1} + \tfrac{3}{2}}{(1-z^{-1})(1+\tfrac{1}{2}z^{-1})}$$

$$= -\tfrac{1}{2} + X_1(z)$$

$$X_1(z) = \frac{-\tfrac{5}{4}z^{-1} + \tfrac{3}{2}}{1-\tfrac{1}{2}z^{-1} - \tfrac{1}{2}z^{-2}}$$

Converting to positive power of z we have

$$X_1(z) = \frac{z(-\tfrac{5}{4} + \tfrac{3}{2}z)}{z^2 - \tfrac{1}{2}z - \tfrac{1}{2}}.$$

$$\frac{X_1(z)}{z} = \frac{(-\tfrac{5}{4} + \tfrac{3}{2}z)}{(z-1)(z+\tfrac{1}{2})}$$

$$\Rightarrow \frac{X_1(z)}{z} = \frac{A_1}{(z-1)} + \frac{A_2}{(z+\tfrac{1}{2})}$$

$$A_1 = (z-1) \cdot \frac{X_1(z)}{z} = \cancel{(z-1)} \cdot \frac{(-\tfrac{5}{4} + \tfrac{3}{2}z)}{\cancel{(z-1)}(z+\tfrac{1}{2})}\Bigg|_{z=1}$$

$$A_1 = \frac{(-\tfrac{5}{4} + \tfrac{3}{2}z)}{(z+\tfrac{1}{2})}\Bigg|_{z=1} = \tfrac{1}{6} \qquad \boxed{A_1 = \tfrac{1}{6}}$$

$$A_2 = (z+\tfrac{1}{2}) \cdot \frac{X_1(z)}{z} = \cancel{(z+\tfrac{1}{2})} \cdot \frac{(-\tfrac{5}{4} + \tfrac{3}{2}z)}{(z-1)\cancel{(z+\tfrac{1}{2})}}\Bigg|_{z=-\tfrac{1}{2}}$$

$$= \frac{-\tfrac{5}{4} + \tfrac{3}{2}z}{(z-1)}\Bigg|_{z=-\tfrac{1}{2}} = \tfrac{4}{3}$$

$$\boxed{A_2 = \tfrac{4}{3}}$$

Scanned with CamScanner

$$\frac{X_1(z)}{z} = \frac{1/6}{(z-1)} + \frac{4/3}{(z+1/2)}$$

$$X_1(z) = \frac{1/6}{(1-z^{-1})} + \frac{4/3}{(1+1/2\,z^{-1})}$$

$$\Rightarrow X(z) = -1/2 + X_1(z)$$

$$= -1/2 + \frac{1/6}{(1-z^{-1})} + \frac{4/3}{(1+1/2\,z^{-1})}$$

Taking inverse Z-transform, we get

$$x(n) = -1/2\,\delta(n) + 1/6\,(1)^n u(n) + 4/3\,(-1/2)^n u(n)$$

$$x(n) = -1/2\,\delta(n) + \frac{1}{3}\left[1/2 + 4(-1/2)^n\right] u(n)$$

* Relationship between Transfer function and difference eqⁿ:-

Consider a discrete time LTI system for which i/p & o/p Satisfy a linear constant-coefficient difference equation of the form.

$$\sum_{k=0}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k)$$

Taking Z transform on both sides, we get

$$\sum_{k=0}^{N} a_k\, Z\{y(n-k)\} = \sum_{k=0}^{M} b_k\, Z\{x(n-k)\}$$

By applying time shift property

$$\sum_{K=0}^{N} a_K z^{-K} Y(z) = \sum_{K=0}^{M} b_K z^{-K} X(z)$$

$$\Rightarrow \quad \frac{Y(z)}{X(z)} = \frac{\sum_{K=0}^{M} b_K z^{-K}}{\sum_{K=0}^{N} a_K z^{-K}}$$

$$\Rightarrow \quad H(z) = \frac{\sum_{K=0}^{M} b_K z^{-K}}{\sum_{K=0}^{N} a_K z^{-K}}.$$

We can say that transfer function of a system described by a linear constant - coefficient difference equation is a ratio of polynomials in z^{-1} and it is referred to as "rational transfer function".

1. Find the transfer function and the impulse response of the system described by the difference equation.

$$y(n) - \tfrac{1}{2} y(n-1) = 2x(n-1)$$

Sol: given $y(n) - \tfrac{1}{2} y(n-1) = 2x(n-1)$

taking Z transforms on both the side

$$Y(z) - \tfrac{1}{2} z^{-1} Y(z) = 2z^{-1} X(z)$$

$$Y(z)\left[1 - \tfrac{1}{2} z^{-1}\right] = 2z^{-1} X(z)$$

$$\frac{Y(z)}{X(z)} = \frac{2z^{-1}}{(1 - \tfrac{1}{2} z^{-1})} = H(z)$$

Taking inverse Z-transform we get

$$h(n) = 2(\tfrac{1}{2})^{n-1} \cdot u(n-1)$$

Scanned with CamScanner

$$H(z) = 2 \cdot \frac{z^{-1}}{(1-\frac{1}{2}z^{-1})}$$ $$\sum_{n=1}^{\infty} \alpha^n = \frac{\alpha}{1-\alpha} = \alpha^n u(n-1)$$ 56

Multiply and divide by $(1/2)$ to get the standard form

$$H(z) = 2\left[\frac{\frac{1}{2}z^{-1}}{1-\frac{1}{2}z^{-1}} (1/2)^{-1}\right]$$

Taking inverse Z transform.

$$h(n) = 2\,(1/2)^n u(n-1)\,(1/2)^{-1}$$

$$h(n) = 2\,(1/2)^{n-1} u(n-1)$$

(2) $y(n) - 1/4\, y(n-1) - 3/8\, y(n-2) = -x(n) + 2x(n-1)$

Taking Z transform on both the sides, we get

$$Y(z) - \tfrac{1}{4} z^{-1} Y(z) - \tfrac{3}{8} z^{-2} Y(z) = -X(z) + 2z^{-1} X(z)$$

$$\Rightarrow Y(z)\left[1 - \tfrac{1}{4}z^{-1} - \tfrac{3}{8}z^{-2}\right] = X(z)\left[-1 + 2z^{-1}\right]$$

$\therefore$ Transfer function $H(z) = \dfrac{Y(z)}{X(z)}$

$$\frac{Y(z)}{X(z)} = \frac{-1+2z^{-1}}{1-\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2}}$$

$$H(z) = \frac{-1+2z^{-1}}{(1+\frac{1}{2}z^{-1})(1-\frac{3}{4}z^{-1})}$$

Powers of numerator $\neq$ Power of denominator there

By use partial fraction expansion to get $h(n)$.

Scanned with CamScanner

Converting to positive powers of z we have

$$H(z) = \frac{-1+2z^{-1}}{(1-\frac{1}{4}z^{-1}-\frac{3}{8}z^{-2})} \times \frac{z^2}{z^2}$$

$$H(z) = \frac{z(-z+2)}{z^2-\frac{1}{4}z-\frac{3}{8}}$$

$$\frac{H(z)}{z} = \frac{-z+2}{(z+\frac{1}{2})(z-\frac{3}{4})}$$

$$\frac{H(z)}{z} = \frac{A_1}{(z+\frac{1}{2})} + \frac{A_2}{(z-\frac{3}{4})}$$

$$A_1 = (z+\tfrac{1}{2}) \cdot \frac{X(z)}{z}\Bigg|_{z=\frac{1}{2}} = \cancel{(z+\tfrac{1}{2})} \cdot \frac{-z+2}{\cancel{(z+\frac{1}{2})}(z-\frac{3}{4})}$$

$$A_1 = \frac{-z+2}{(z-\frac{3}{4})}\Bigg|_{z=-\frac{1}{2}} = -2. \qquad \boxed{A_1 = -2}$$

$$A_2 = (z-\tfrac{3}{4}) \cdot \frac{X(z)}{z}\Bigg|_{z=\frac{3}{4}} = \frac{-z+2}{(z+\frac{1}{2})}\Bigg|_{z=\frac{3}{4}}$$

$$\boxed{A_2 = 1}$$

$$\therefore \frac{H(z)}{z} = \frac{-2}{(z+\frac{1}{2})} + \frac{1}{(z-\frac{3}{4})}$$

$$H(z) = \frac{-2z}{(z+\frac{1}{2})} + \frac{z}{(z-\frac{3}{4})}$$

$$= \frac{-2}{(1+\frac{1}{2}z^{-1})} + \frac{1}{(1-\frac{3}{4}z^{-1})}$$

Scanned with CamScanner

taking inverse Z-transform, we get

$$h(n) = -2(-1/2)^n u(n) + 1(3/4)^n u(n)$$

$$= \left[-2(-1/2)^n + (3/4)^n\right] u(n)$$

* Stability and Causality :-

- Characteristics of system such as Casuality and Stability can be determined from pole zero pattern and the ROC of transfer function $H(z)$.
- For a system to be casual, its impulse response $h(n)$ must be equal to zero for $n<0$ i.e $h(n)=0$; $n<0$

  $\therefore$ $h(n)$ must be a right sided.
- if the system is casual, then ROC for $H(z)$ will be outside the outermost pole.
- for a system to be stable, its impulse response $h(n)$ must be absolutely summable i.e $\sum_{k=-\infty}^{\infty} |h(n)| < \infty$.

Alternatively, a casual system is stable if the poles of $H(z)$ lies inside the unit circle in the Z plane.

- For a system that is both stable and casual, ROC must include unit circle and it must be outside the outer most pole. i.e all the poles should lie inside the unit circle in Z-plane

Scanned with CamScanner

59

1. Determine whether the system described below is Casual and stable.

a. $H(z) = \dfrac{2z+1}{z^2+z-5/16}$

$$H(z) = \frac{2(z+1/2)}{(z+5/4)(z-1/4)}$$

∴ poles of the above expression. $z = -5/4$ $z = 1/4$

$z = -5/4 = -1.25$ hence one of the pole lies outside the unit circle there by system is not both casual and stable.

2. the System function of the LTI system is given as.

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$$

Specify the ROC of H(z) and determine unit sample response h(n) for the following conditions:-

1. Stable system 2. Casual system

Sol: $H(z) = \dfrac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}}$

Convert powers of z in H(z) to positive values.

$$H(z) = \frac{3-4z^{-1}}{1-3.5z^{-1}+1.5z^{-2}} \times \frac{z^2}{z^2}$$

$$= \frac{(3z-4)z}{z^2-3.5z+1.5}$$

$$\frac{H(z)}{z} = \frac{3z-4}{z^2-3.5z+1.5}$$

$$= \frac{3z-4}{(z-3)(z-0.5)}$$

Scanned with CamScanner

$$\frac{H(z)}{z} = \frac{A_1}{(z-3)} + \frac{A_2}{(z-0.5)}$$

$$\Rightarrow A_1 = (z-3) \cdot \frac{H(z)}{z}\Big|_{z=3} = \cancel{(z-3)} \cdot \frac{3z-4}{\cancel{(z-3)}(z-0.5)}\Big|_{z=3} = \frac{3z-4}{(z-0.5)}\Big|_{z=3}$$

$$\boxed{A_1 = 2}$$

$$A_2 = (z-0.5)\frac{H(z)}{z}\Big|_{z=0.5} = \frac{3z-4}{(z-3)}\Big|_{z=0.5} = 1.$$

$$\boxed{A_2 = 1}$$

$$\therefore \frac{H(z)}{z} = \frac{2}{(z-3)} + \frac{1}{(z-0.5)}$$

$$H(z) = \frac{2z}{(z-3)} + \frac{z}{(z-0.5)} = \frac{2}{(1-3z^{-1})} + \frac{1}{(1-0.5z^{-1})}$$

$\therefore |z| < 3$ and $|z| > 0.5 \rightarrow$ ROC.

1. find $h(n)$ for stable system.

ROC of $H(z)$ must have a unit circle

$\therefore$ System has one pole at $z = 3$ and second pole at $z = 0.5$ hence ROC: $0.5 < |z| < 3$

$\therefore |z| = 1$ is included inside the the range of ROC.

$\therefore$ IZT of $\left\{\frac{1}{1-0.5z^{-1}}\right\} = a^n u(n) \rightarrow$ ROC: $|z| > 0.5$

hence it is a positive time sequence $= (0.5)^n u(n)$

IZT $\left\{\frac{2}{(1-3z^{-1})}\right\}$; ROC $= |z| < 3 \rightarrow$ negative time sequence of the form $-b^n u(-n-1)$

Scanned with CamScanner

$\therefore h(n) = IZT\{H(z)\} = -2(3)^n u(-n-1) + (0.5)^n u(n)$

2. find h(n) for Casual System:-

- for a Casual System ROC for H(Z) should be outside the outer most pole (exterior of the Circle will be ROC region).

- for this Condition to satisfy $|z| > 3$.

$$\therefore h(n) = IZT\{H(z)\} = IZT\left\{\frac{2}{1-3z^{-1}}\right\} + IZT\left\{\frac{1}{1-0.5z^{-1}}\right\}$$

$$= 2(3)^n u(n) + (0.5)^n u(n)$$

$$= \left[2(3)^n + (0.5)^n\right] u(n).$$

* Inverse System :-

- For a System having impulse response h(n), its inverse System impulse response $h^{-1}(n)$ is such that.

$$h^{-1}(n) * h(n) = \delta(n)$$

Taking Z-transform on both the sides, we get.

$$H^{-1}(z) \cdot H(z) = 1.$$

$$\therefore H^{-1}(z) = \frac{1}{H(z)}$$

$\therefore$ the transfer function of an inverse System is the inverse of the transfer function of the System i.e Zeros of H(z) are the poles of $H^{-1}(z)$ and poles of H(Z) are the Zeros of $H^{-1}(z)$.

- For a System $H^{-1}(z)$ to be both stable and casual its ROC must include a Unit Circle and it must be

Scanned with CamScanner

Outside the Outermost pole.

- $H^{-1}(z)$ is both stable and casual if all of its poles are inside the unit circle.
- A system with all its poles and zeros inside the unit circle in the z-plane is refered as "minimum phase system".

Scanned with CamScanner

② A difference equation of the System is given as,

$$y(n) - y(n-1) + \tfrac{1}{4}y(n-2) = x(n) + \frac{1}{4}x(n-1) - \frac{1}{8}x(n-2)$$

Determine the transfer function of inverse system. Check whether the inverse system is casual and stable.

given: $y(n) - y(n-1) + \tfrac{1}{4}y(n-2) = x(n) + \frac{1}{4}x(n-1) - \frac{1}{8}x(n-2)$

Taking z transform of the given equation.

$$Y(z) - Y(z)\cdot z^{-1} + \frac{1}{4}z^{-2}Y(z) = X(z) + \frac{1}{4}z^{-1}X(z) - \frac{1}{8}z^{-2}X(z)$$

$$Y(z)\left[1 - z^{-1} + \tfrac{1}{4}z^{-2}\right] = X(z)\left[1 + \tfrac{1}{4}z^{-1} - \frac{1}{8}z^{-2}\right]$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{(1 + \tfrac{1}{4}z^{-1} - \tfrac{1}{8}z^{-2})}{(1 - z^{-1} + \tfrac{1}{4}z^{-2})}$$

Transfer function of the inverse system is given by

$$H^{-1}(z) = \frac{1}{H(z)} = \frac{1 - z^{-1} + \tfrac{1}{4}z^{-2}}{1 + \tfrac{1}{4}z^{-1} - \tfrac{1}{8}z^{-2}}$$

$$= \frac{(z - \tfrac{1}{2})(z - \tfrac{1}{2})}{(z - \tfrac{1}{4})(z + \tfrac{1}{2})}$$

$\therefore$ poles $z = \tfrac{1}{4}$, $z = -\tfrac{1}{2}$

hence both the poles are inside the unit circle hence it is casual and stable system.

* **Unilateral z transform :-**

Unilateral or one sided z-transform is defined

Scanned with CamScanner

for $n \geq 0$ i.e non negative values of n.

- the unilateral z-transform $X(z)$ of a sequence $x(n)$ is defined as.

$$X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

the Summation is carried out only for non negative values of n. We denote the relationship between $x(n)$ and $X(z)$ as.

$$x(n) \xleftrightarrow{Zu} X(z)$$

The unilateral and bilateral z-transform are equivalent for casual signals. Both transform satisfy all properties except time shift property.

Consider $x(n) \xleftrightarrow{Zu} X(z)$

Let $y(n) = x(n-1)$

then unilateral z-transform of $y(n)$

$$Y(z) = \sum_{n=0}^{\infty} y(n) . z^{-n}.$$

$$= \sum_{n=0}^{\infty} x(n-1) z^{-n}.$$

$$= x(-1) + \sum_{n=1}^{\infty} x(n-1) z^{-n}.$$

Substituting $n-1=m$, we get

$$= x(-1) + \sum_{m=0}^{\infty} x(m) z^{-(m+1)}$$

Scanned with CamScanner

$$Y(z) = x(-1) + z^{-1}\sum_{m=0}^{\infty} x(m) z^{-m}.$$

$$Y(z) = x(-1) + z^{-1} X(z)$$

Similarly, let $w(n) = x(n-2)$

$$W(z) = \sum_{n=0}^{\infty} w(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} x(n-2) z^{-n}.$$

$$W(z) = x(-2) + x(-1) z^{-1} + \sum_{n=2}^{\infty} x(n-2) z^{-n}.$$

Substituting $n-2 = m$.

$$W(z) = x(-2) + x(-1) z^{-1} + \sum_{m=0}^{\infty} x(m) z^{-(m+2)}$$

$$W(z) = x(-2) + x(-1) z^{-1} + z^{-2} X(z)$$

(1) Determine the one-sided or unilateral z transform for the signal $x(n) = \{1, 2, 5, 7, 0, 1\}$

By defination $X(z) = \sum_{n=0}^{\infty} x(n). z^{-n}.$

$$X(z) = \sum_{n=0}^{5} x(n). z^{-n}.$$

$$= x(0) z^{-0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3} + x(4) z^{-4} + x(5) z^{-5}$$

$$= 1 + 2z^{-1} + 5z^{-2} + 7z^{-3} + z^{-5}$$

Scanned with CamScanner

(2) Find the unilateral Z-transform for the signal.

a. $x(n) = \alpha^n u(n)$

b. $y(n) = x(n-2)$ where $x(n) = \alpha^n$.

a. By using defination of unilateral Z-transform.

$$x(z) = \sum_{n=0}^{\infty} x(n) z^{-n}.$$

$$X(Z) = \sum_{n=0}^{\infty} \alpha^n u(n) z^{-n}.$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n}. - \sum_{n=0}^{\infty} (\alpha z^{-1})^n.$$

$$= \frac{1}{1-\alpha z^{-1}}$$

b. $x(n) = \alpha^n$. $\therefore X(Z) = \dfrac{1}{1-\alpha z^{-1}}$

given $y(n) = x(n-2)$

$$Y(Z) = x(-2) + x(-1) z^{-1} + z^{-2} X(Z)$$

$$x(-2) = \alpha^{-2}, \quad x(-1) = \alpha^{-1}$$

$$\therefore Y(Z) = \alpha^{-2} + \alpha^{-1} z^{-1} + \frac{z^{-2}}{1-\alpha z^{-1}}.$$

Scanned with CamScanner

③ Solve the difference equation

$y(n) + 3y(n-1) = x(n)$ with $x(n) = u(n)$ and initial Condition $y(-1) = 1$.

Sol: given: $y(n) + 3y(n-1) = x(n)$

Taking unilateral z-transform on both sides, we get

$$Y(z) + 3\left[y(-1) + z^{-1}Y(z)\right] = X(z) \quad -① $$

given $x(n) = u(n) \Rightarrow X(z) = \frac{1}{1-z^{-1}}$, $y(-1) = 1$.

Substituting in eq ① we get

$$Y(z) + 3\left[1 + z^{-1}Y(z)\right] = \frac{1}{1-z^{-1}}$$

$$Y(z)\left[1+3z^{-1}\right] + 3 = \frac{1}{1-z^{-1}}$$

$$Y(z) + \frac{3}{(1+3z^{-1})} = \frac{1}{1-z^{-1}\,(1+3z^{-1})}$$

$$Y(z) = -\frac{3}{(1+3z^{-1})} + \frac{1}{(1+3z^{-1})(1-z^{-1})}$$

Using partial fraction method to solve the expression.

$$Y_1(z) = \frac{1}{(1+3z^{-1})(1-z^{-1})} \times \frac{z^2}{z^2} = \cancel{\frac{A_1 z^2}{(1+3z)} + \frac{A_2 z^2}{(z-1)}}$$

$$\frac{Y_1(z)}{z} = \cancel{\frac{A_1 z}{(z+3)} + \frac{A_2 z}{(z-1)}} = \frac{z}{(z+3)(z-1)}$$

$$\frac{Y_1(z)}{z} = \frac{A_1}{(z+3)} + \frac{A_2}{(z-1)}$$

Scanned with CamScanner

$$A_1 = (z+3) \cdot \frac{Y_1(z)}{z}\Big|_{z=-3} = (z+3) \cdot \frac{z}{(z+3)(z-1)}\Big|_{z=-3}$$

$$= \frac{z}{(z-1)}\Big|_{z=-3} = 3/4 \quad \boxed{A_1 = 3/4}$$

$$A_2 = (z-1) \cdot \frac{Y_1(z)}{z}\Big|_{z=1} = \frac{z}{(z-1)}\Big|_{z=1} = 1/4$$

$$\therefore Y(z) = \frac{-3}{(1+3z^{-1})} + \frac{3/4}{(1+3z^{-1})} + \frac{1/4}{(1-z^{-1})}$$

$$= \frac{-9/4}{(1+3z^{-1})} + \frac{1/4}{(1-z^{-1})}$$

Taking inverse Z-transform, we get

$$y(n) = -9/4\,(-3)^n u(n) + 1/4\, u(n)$$

$$= \left[1/4 - 9/4\,(-3)^n\right] u(n).$$

(4) Solve the following difference equation for the given initial conditions and input $y(n) - 1/9\, y(n-2) = x(n-1)$ with $y(-1) = 0$, $y(-2) = 1$ and $x(n) = 3u(n)$.

Sol: $y(n) - 1/9\, y(n-2) = x(n-1)$

Taking unilateral Z-transform on both sides, we get

$$Y(z) - 1/9\left[y(-2) + y(-1)z^{-1} + z^{-2}Y(z)\right] = x(-1) + z^{-1}X(z)$$

Substitute values for $y(-2)=1$, $y(-1)=0$, $x(-1)=0$

$$Y(z) - 1/9\left[1 + z^{-2}Y(z)\right] = z^{-1}X(z)$$

$$Y(z)\left[1-\tfrac{1}{9}z^{-2}\right] = \frac{1}{9} + z^{-1}\left[\frac{3}{1-z^{-1}}\right]$$

$$Y(z) = \left[\frac{1}{9} + \frac{3z^{-1}}{(1-z^{-1})}\right]\cdot\frac{1}{(1-\tfrac{1}{9}z^{-2})}$$

$$= \frac{(3z^{-1})9 + (1-z^{-1})}{9(1-z^{-1})(1-\tfrac{1}{9}z^{-2})}$$

$$= \frac{(1-z^{-1}+27z^{-1})}{9(1-z^{-1})(1-\tfrac{1}{9}z^{-2})}$$

$$= \frac{\tfrac{1}{9}(1+26z^{-1})}{(1-\tfrac{1}{9}z^{-2})(1-z^{-1})}$$

$$= \frac{\tfrac{1}{9}(1+26z^{-1})}{(1+\tfrac{1}{3}z^{-1})(1-\tfrac{1}{3}z^{-1})(1-z^{-1})}$$

By partial fraction expansion, we get

$$Y(z) = \frac{\tfrac{1}{9}(z+26)z^{2}}{(z+\tfrac{1}{3})(z-\tfrac{1}{3})(z-1)}$$

$$\frac{Y(z)}{z} = \frac{\tfrac{1}{9}(z+26)z}{(z+\tfrac{1}{3})(z-\tfrac{1}{3})(z-1)}$$

$$\therefore \frac{Y(z)}{z} = \frac{A_1}{(z+\tfrac{1}{3})} + \frac{A_2}{(z-\tfrac{1}{3})} + \frac{A_3}{(z-1)}$$

$$A_1 = (z+\tfrac{1}{3})\cdot\frac{Y(z)}{z}\bigg|_{z=-1/3} = \frac{\tfrac{1}{9}(z+26)z}{\cancel{(z+\tfrac{1}{3})}(z-\tfrac{1}{3})(z-1)}\bigg|_{z=-1/3}$$

$$\boxed{A_1 = -77/72}$$

Scanned with CamScanner

$$A_2 = (z - 1/3)\,\frac{Y(z)}{z}\Big|_{z=1/3} = \frac{1/9\,(z+26)\,z}{(z+1/3)\,(z-1)}\Big|_{z=1/3}$$

$$\boxed{A_2 = -79/36}$$

$$A_3 = (z-1)\cdot\frac{Y(z)}{z}\Big|_{z=1} = \frac{1/9\,(z+26)\,z}{(z-1/3)\,(z+1/3)}\Big|_{z=1}$$

$$\boxed{A_3 = 27/8}$$

$$\Rightarrow Y(z) = \frac{-77/72 \cdot z}{(z+1/3)} + \frac{-79/36 \cdot z}{(z-1/3)} + \frac{27/8\, z}{(z-1)}$$

$$= \frac{-77/72}{(1+1/3\,z^{-1})} + \frac{-79/36}{(1-1/3\,z^{-1})} + \frac{27/8}{(1-z^{-1})}$$

Taking inverse z-transform, we get

$$y(n) = -\frac{77}{72}(-1/3)^n u(n) - \frac{79}{36}(1/3)^n u(n) + \frac{27}{8}u(n)$$

$$y(n) = \left[-77/72\,(-1/3)^n - 79/36\,(1/3)^n u(n) + 27/8\right] u(n)$$

— x —

Scanned with CamScanner