Unit-4

Fourier Representation of Signal.

- Convolution Sum and Convolution integral are the convenient way to find the response of an LTI system if its impulse response is known.
- In this chapter we will see an alternate representation for signals and System where we represent a Signal as a "weighted Superposition of Complex Sinusoidal"
- In case such signal is applied to a LTI system, then System o/p will be a weighted Superposition of complex Sinusoidal.
- this representation is called "Fourier representation".
- this concept is a contribution of "Joseph Fourier".

* Fourier representation for Signal classes:-

Depending on the periodic nature of the signal there are 4 distinct Fourier representation

1. Periodic signals - Fourier Series.
2. Nonperiodic Signals - Fourier transforms.

Time property	Periodic	Non periodic
1. Continuous	Fourier Series (FS)	Fourier transform (FT)
2. Discrete	Discrete Fourier Series (DTFS)	Discrete time Fourier transform (DTFT)

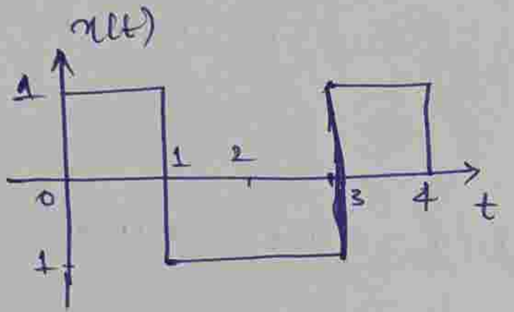
* Orthogonality of Complex Sinusoidal Signals:-

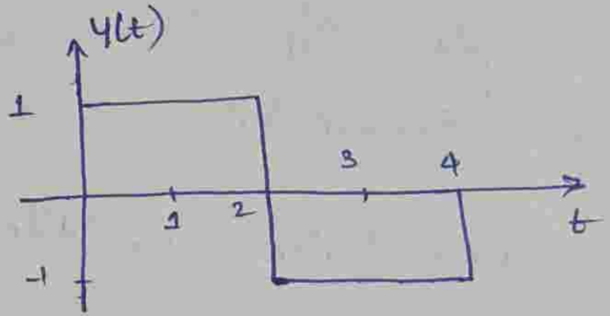
Lets consider two continuous - time signal $x(t)$ and $y(t)$ these two signals are said to be orthogonal over a period of time / interval (a, b) if

$$\int_a^b x(t)\, y^*(t)\, dt = 0.$$

where $y^*(t)$ is the complex conjugate of $y(t)$.

if $z = a + ib$ its conjugate will by $\bar{z} = a - ib$



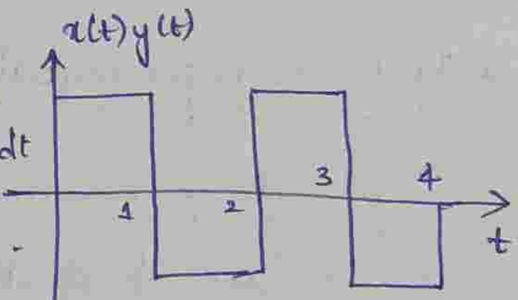


time interval $(0, 4)$

$y(t) = \text{real}$ $\therefore y^*(t) = y(t)$

$$\int_0^4 x(t) \cdot y(t) = \int_0^1 1\,dt - \int_1^2 1\,dt + \int_2^3 1\,dt - \int_3^4 1\,dt$$

$$= 0.$$



$\therefore$ $x(t)$ and $y(t)$ are orthogonal over the interval $(0, 4)$

Similarly two discrete time signal $\phi_k(n)$ and $\phi_m(n)$ are said to be orthogonal over the interval (N_1, N_2)

if $$\sum_{n=N_1}^{N_2} \phi_k(n)\, \phi_m^*(n) = A_k \quad ; k = m.$$
$$= 0 \quad ; k \neq m.$$

A_k = Constant.

* Discrete time Periodic Signal - Discrete time fourier Series (DTFS). [frequence domain representation of time domain Signal]

A discrete time fourier series representation of a periodic signal $x(n)$ is given by.

$$x(n) = \sum_{K=\langle N \rangle} X[K]\, e^{jK\Omega_0 n} \quad - (1)$$

$$X(K) = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega_0 n} \quad - (2)$$

- when the i/p is discrete and periodic i.e

$$x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$$

[periodic] [frequency domain representation of periodic signal]

- where $X[K]$ is the Discrete time fourier Series Co-efficient. which Specifies the decomposition of the signal $x(n)$ into a sum of N harmonically related complex exponentials.

- equation (1) is know as Synthesis equation
  (2) is know as analysis equation.

$$X(K) = \frac{1}{N} \sum_{n=N} x(n)\, e^{-jK\Omega_0 n} \rightarrow \text{analysis}.$$

using this equation we can generate fouries Series co-efficients in frequency domain.

where K = index for co-efficients in frequency domain
N = fundamental period of discrete time i/p signal.

$\Omega_0 = \frac{2\pi}{N}$ = fundamental frequency.

eq ① $x(n) = \sum_{K=\langle N\rangle} X(K)\, e^{jK\Omega_0 n}$

this is used to obtain $x(n)$ from $X[K]$

the value of k and n are arbitrary as both $x(n)$ & $X[K]$ are periodic with period N.

* **Properties of Discrete time Fourier Series [DTFS]**

1. **Linearity** :

If $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ and $y(n) \xleftrightarrow{DTFS,\ \Omega_0} Y[K]$

then $Z(n) = ax(n) + by(n) \xleftrightarrow{DTFS,\ \Omega_0} aX[K] + bY[K]$

Proof: Assume both $x(n)$ and $y(n)$ are having same fundamental period $N = \frac{2\pi}{\Omega_0}$

By defination $X(k) = \frac{1}{N} \sum_{n=\langle N\rangle} x(n)\, e^{-jK\Omega_0 n}$

$$Y[K] = \frac{1}{N} \sum_{n=\langle N\rangle} y(n)\, e^{-jK\Omega_0 n}$$

$$\therefore Z[K] = \frac{1}{N} \sum_{n=\langle N\rangle} Z(n)\, e^{-jK\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} [ax(n) + by(n)]\, e^{-jK\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N\rangle} ax(n)\, e^{-jK\Omega_0 n} + \frac{1}{N} \sum_{n=\langle N\rangle} by(n)\, e^{-jK\Omega_0 n}$$

$$= aX[K] + bY[K]$$

2. Time Shift :

If $x(n) \xleftrightarrow{DTFS, \Omega_0} X[K]$ then

$$w(n) = x(n-n_0) \xleftrightarrow{DTFS, \Omega_0} W[k] = e^{-jK\Omega_0 n_0} X[K]$$

Proof : Using the defination.

$$X[K] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jK\Omega_0 n}$$

$$W[K] = \frac{1}{N} \sum_{n=\langle N \rangle} w(n) e^{-jK\Omega_0 n} \qquad w(n) = x(n-n_0)$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n-n_0) e^{-jK\Omega_0 n}$$

Put $n - n_0 = m$, then $n = m + n_0$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} x(m) \cdot e^{-jK\Omega_0 (m+n_0)}$$

$$= \frac{1}{N} \sum_{m=\langle N \rangle} x(m) \cdot e^{-jK\Omega_0 m} \cdot e^{-jK\Omega_0 n_0}$$

$$= X[K] e^{-jK\Omega_0 n_0}$$

3. Frequency Shift :

If $x(n) \xleftrightarrow{DFTS, \Omega_0} X[K]$ then.

$$g(n) = e^{jK_0\Omega_0 n} x(n) \xleftrightarrow{DFTS, \Omega_0} G[K] = X[K-K_0]$$

Proof : $$G[K] = \frac{1}{N} \sum_{n=\langle N \rangle} g(n) e^{-jK\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} e^{jK_0\Omega_0} x(n) e^{-jK\Omega_0 n}$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-j(K-K_0)\Omega_0 n}.$$

$$= X[K-K_0]$$

4: <u>Scaling:</u>

- We know that Scaling operation of discrete time Signal discards information.

- Due to loss of information it is not possible to express the DTFS of Scaled signal in terms of DTFS of original Signal.

Consider a periodic discrete time signal $x(n)$ with fundamental period N, Such that

$$x(n) = 0 \quad ; \text{ unless } \frac{n}{p} \text{ is integer.}$$

then $z(n) = x(pn)$ has a fundamental period N/p

In this case if $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ then.

$$z(n) = x(pn) \xleftrightarrow{DTFS,\ p\Omega_0} Z[K] = pX[K]$$

- Scaling operation changes the harmonic space from Ω_0 to $p\Omega_0$ and amplifies the DTFS co-efficients by p.

5. <u>Convolution:-</u>

If $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ and $y(n) \xleftrightarrow{DTFS,\ \Omega_0} Y[K]$

then $z(n) = x(n) \circledast y(n) \xleftrightarrow{DTFS,\ \Omega_0} Z[K] = N.[X(K).Y(K)]$.

↑ periodic Convolution.

Proof : $X[K] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jK\Omega_0 n}$

$$y(K) = \frac{1}{N} \sum_{n=\langle N \rangle} y(n)\, e^{-jk\Omega_o n}$$

$$z[K] = \frac{1}{N} \sum_{n=\langle N \rangle} z(n)\, e^{-jk\Omega_o n}.$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} [x(n) \circledast y(n)]\, e^{-jk\Omega_o n}.$$

By using defination of Convolution, we have.

$$z[K] = \frac{1}{N} \sum_{n=\langle N \rangle} \Big[\sum_{l=\langle N \rangle} x(l)\, y(n-l) \Big]\, e^{-jk\Omega_o n}.$$

Changing the order of summation, we get

$$z[K] = \frac{1}{N} \Big[\sum_{l=\langle N \rangle} x(l) \sum_{n=\langle N \rangle} y(n-l)\, e^{-jk\Omega_o n} \Big]$$

$n-l=m$ then $n=m+l$.

$$z[K] = \frac{1}{N} \Big[\sum_{l=\langle N \rangle} x(l) \sum_{m=\langle N \rangle} y(m)\, e^{-jk\Omega_o (m+l)} \Big]$$

$$= \frac{1}{N} \Big[\sum_{l=\langle N \rangle} x(l) \cdot e^{-jk\Omega_o l} \sum_{m=\langle N \rangle} y(m)\, e^{-jk\Omega_o m} \Big]$$

$$= \frac{1}{N} \Big[N\, X[K] \cdot N\, Y[K] \Big]$$

$$= N \big[X[K]\, Y[K] \big]$$

Convolution in time domain is transformed to multiplication of DTFS Co-efficient.

6. Modulation :

If $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ and $y(n) \xleftrightarrow{DTFS,\ \Omega_0} Y[K]$

then $z(n) = x(n)y(n) \xleftrightarrow{DTFS,\ \Omega_0} Z[K] = X[K] \circledast Y[K]$

Proof : We have $Z[K] = \frac{1}{N} \sum_{n=\langle N\rangle} z(n)\, e^{-jK\Omega_0 n}$.

$$Z[K] = \frac{1}{N} \sum_{n=\langle N\rangle} [x(n)\, y(n)]\, e^{-jK\Omega_0 n}.$$

from defination of $x(n) = \sum_{l=\langle N\rangle} X[l]\, e^{jl\Omega_0 n}$.

$$Z[K] = \frac{1}{N} \sum_{n=\langle N\rangle} \left[\sum_{l=\langle N\rangle} X[l]\, e^{jl\Omega_0 n} \right] y(n)\, e^{-jK\Omega_0 n}.$$

changing the order of summation :

$$Z[K] = \sum_{l=\langle N\rangle} X[l] \cdot \frac{1}{N} \sum_{n=\langle N\rangle} y(n)\, e^{-j(K-l)\Omega_0 n}.$$

$$= \sum_{l=\langle N\rangle} X[l]\ Y[K-l]$$

$$Z[K] = X[K] \circledast Y[K]$$

7. Parseval's theorem :-

If $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ then.

$$\frac{1}{N} \sum_{n=\langle N\rangle} |x(n)|^2 = \sum_{K=\langle N\rangle} |X[K]|^2.$$

this is the average power of a periodic signal with period N. [LHS]

$$\text{Power} = \frac{1}{N} \sum_{n=\langle N \rangle} |x(n)|^2$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \cdot x^*(n)$$

$$= \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \left[\sum_{K=\langle n \rangle} X^*[K] \, e^{-jK\Omega_0 n} \right]$$

Changing the order of summation

$$= \sum_{K=\langle N \rangle} X^*[K] \cdot \frac{1}{N} \sum_{n=\langle N \rangle} x(n) \, e^{-jK\Omega_0 n}$$

$$= \sum_{K=\langle N \rangle} X^*[K] \cdot X[K]$$

$$= \sum_{K=\langle N \rangle} |X[K]|^2$$

the Sequence $|X[K]|^2$ for $K = 0, 1, 2, \ldots N-1$ is the distribution of power as a function of frequency and its called "Power density Spectrum" of the Signal $x(n)$

8. <u>Duality</u> :-

If $x(n) \xleftrightarrow{DTFS,\ \Omega_0} X[K]$ then.

$$X[n] \xleftrightarrow{DTFS,\ \Omega_0} \frac{1}{N} x(-K)$$

Proof: We have $x(n) = \sum_{k=\langle N \rangle} X[k]\, e^{j\Omega_0 kn}$

Replace n by −n. we get.

$$x(-n) = \sum_{k=\langle N \rangle} X[k]\, e^{jk\Omega_0(-n)}$$

Replace n by k and k by n.

$$x(-k) = \sum_{n=\langle N \rangle} X[n]\, e^{-jk\Omega_0 n}$$

Multiply both sides by 1/N.

$$\frac{1}{N} x(-k) = \frac{1}{N} \sum_{n=\langle N \rangle} X[n]\, e^{-jk\Omega_0 n}.$$

$$= \text{DTFS}\{X(n)\}$$

comparing with $X(k) = \frac{1}{N} \sum_{n=\langle N \rangle} x[n]\, e^{-jk\Omega_0 n}$.

$$X(n) \xleftrightarrow{\text{DTFS},\, \Omega_0} \frac{1}{N} x(-k)$$

9. Symmetry :- If $x(n) \xleftrightarrow{\text{DTFS},\, \Omega_0} X[k]$ then $x(n)$ real

$$x(n) \text{ real} \xleftrightarrow{\text{DTFS},\, \Omega_0} X^*[k] = X[-k]$$

$$x(n) \text{ img} \xleftrightarrow{\text{DTFS},\, \Omega_0} X^*[k] = -X[-k]$$

$$x(n) = x_e(n) + x_o(n)$$

$$x_e(n) \text{ real} \xleftrightarrow{\text{DTFS},\, \Omega_0} \text{Img}\{X[k]\} = 0$$

$$x_o(n)\ \cancel{\text{real}} \xleftrightarrow{\text{DTFS},\, \Omega_0} \text{Re}\{X[k]\} = 0.$$

1) Determine the spectra of the signal $x(n) = \cos\frac{\pi}{3}n$

Sol:- We know that $x(n) = \cos\Omega_0 n$ is periodic if Ω_0 is integral multiple of $\frac{2\pi}{N}$ where N is the fundamental period i.e $\Omega_0 = \frac{2\pi}{N} \cdot m$.

By Comparing $x(n) = \cos\frac{\pi}{3}n$ with $x(n) = \cos\Omega_0 n$

$$\Omega_0 = \frac{\pi}{3} = \frac{2\pi}{6} \cdot 1 = \frac{2\pi}{N} \cdot m$$

where fundamental period $N = 6$.

we know that $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n) e^{-jk\Omega_0 n}$ — analysis eq^n

$$x(n) = \cos\pi/3\, n = \frac{1}{2} e^{j\pi/3 n} + \frac{1}{2} e^{-j\pi/3 n} \quad -(1)$$

Comparing equation (1) with the equation

$$x(n) = \sum_{K=\langle N \rangle} X[K] e^{jk\Omega_0 n} \rightarrow \text{Synthesis eq}^n$$

eq (1) can be written as

$$x(n) = \frac{1}{2} e^{j(1)\pi/3 n} + \frac{1}{2} e^{j(-1)\pi/3 n}$$

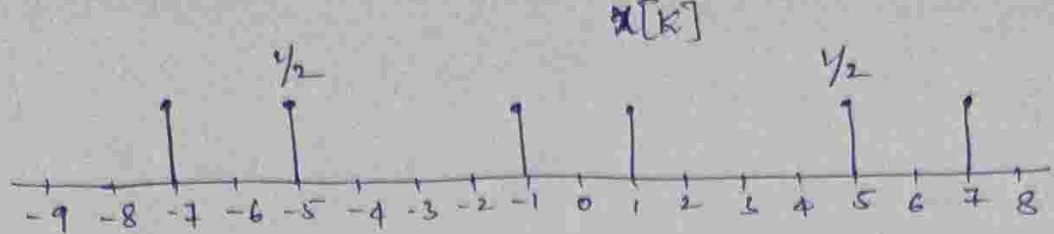
$$X[1] = \frac{1}{2} \quad X[-1] = \frac{1}{2}$$

Since DTFS $X[K]$ forms a periodic signal of period N, we can write

$$\ldots X[-11] = X[-5] = X[1] = X[7] = X[13] = \frac{1}{2}$$

$$\ldots X[-7] = X[-1] = X[5] = X[11] = X[17] = \frac{1}{2}$$

and other $X[K]$ are equal to zero.



② Evaluate the DTFS representation for the signal.

$$x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1.$$

Sketch the magnitude and phase spectrum.

Sol: Given $x(n) = \sin\left(\frac{4\pi}{21}n\right) + \cos\left(\frac{10\pi}{21}n\right) + 1.$

1. ∴ angular frequency $\Omega_0 = \left(\frac{4\pi}{21}, \frac{10\pi}{21}\right)$

Since there are two different Ω_0 we need to find the gcd of Ω_{01}, Ω_{02}.

$\Rightarrow \Omega_0 = \gcd(\Omega_{01}, \Omega_{02}) = \gcd\left(\frac{4\pi}{21}, \frac{10\pi}{21}\right)$

$\Rightarrow \Omega_0 = \frac{2\pi}{21}$ ∴ fundamental period $N = 21$.

2. Arrange given signal in terms of x(n)

$$\Rightarrow x(n) = \frac{e^{j4\pi/21 n} - e^{-j4\pi/21 n}}{2j} + \frac{e^{j10\pi/21 n} + e^{-j10\pi/21 n}}{2} + 1.$$

Arrange the terms in the form of a sequence in order to obtain X[K] co-efficients as.

$$x(n) = \sum_{K=\langle N \rangle} X[K]\, e^{jK\Omega_0 n} \quad -①$$

$$\Rightarrow x(n) = \frac{1}{2j} e^{j(2)2\pi/21 n} - \frac{1}{2j} e^{j(-2)2\pi/21 n} + \frac{1}{2} e^{j(5)(2\pi/21)n} + \frac{1}{2} e^{j(-5)(2\pi/21)n} + 1$$

$\therefore$ Comparing the sequence with eq (1) we have. (9)

$x[0] = 1 \quad x[-2] = -\frac{1}{2j} \quad x[2] = \frac{1}{2j} \quad x[5] = \frac{1}{2} \quad x[-5] = \frac{1}{2}$

4. plot the magnitude and phase spectrum

1. magnitude : $|x[K]|$

$|x[2]| - \frac{1}{2j} \times \frac{j}{j} = \frac{j}{-2} = -0.5j$ this is of the form.

$|a+bj| = \sqrt{a^2+b^2} \Rightarrow |0-0.5j| = \sqrt{0^2+(0.5)^2} = 0.5$

$\therefore \boxed{|x[2]| = \frac{1}{2}}$

$|x[-2]| = -\frac{1}{2j} \times \frac{j}{j} = \frac{-j}{-2} = 0.5j \Rightarrow 0+0.5j$

$\therefore |x[-2]| - \sqrt{0^2+0.5^2} = 0.5$

$\boxed{|x[-2]| = \frac{1}{2}}$

2. phase : $\angle(a+ib) = \tan^{-1}(b/a)$

$\therefore x[-2] = 0+0.5j \quad \Rightarrow \angle x[-2] = \tan^{-1}(0.5/0) = \tan^{-1}(\infty) = \pi/2$

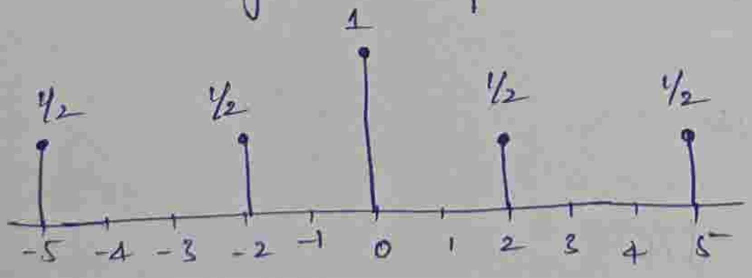
$x[2] = 0-0.5j \quad \angle x[2] = \tan^{-1}(0.5/0) = \tan^{-1}(\infty) = -\pi/2$

$x[5] = \frac{1}{2}+0j \quad \angle x[5] = \tan^{-1}(0/0.5) = \tan^{-1}(0) = 0$

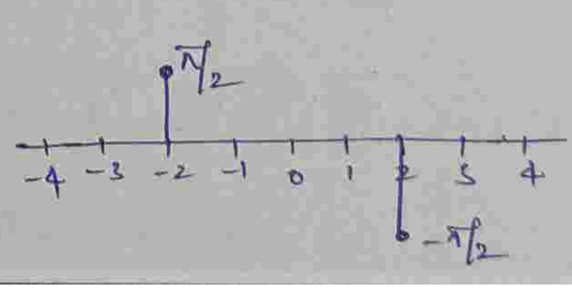
$x[-5] = \frac{1}{2}+0j \quad \angle x[-5] = 0$

$x[0] = 1+0j \quad \angle x[0] = 0$

Magnitude Spectrum



Phase spectrum



(3) $x(n) = \cos\left(\frac{6\pi}{13} n + \pi/6\right)$

here given func" is a single function but there is a phase shift of $\pi/6$

Step 1: find the fundamental period.

$\therefore \Omega_0 = \frac{6\pi}{13}$ this has to be in the form of $\frac{2\pi}{N} \cdot m$.

$\therefore \Omega_0 = \frac{2\pi}{13} \cdot 3$ $\therefore$ fundamental period $N = 13$.

Step 2: express in terms of synthesis equation

$$x(n) = \sum_{k=\langle N\rangle} x[k]\, e^{jk\Omega_0 n}$$

$$x(n) = \frac{e^{j\left(\frac{6\pi}{13}n + \pi/6\right)} + e^{-j\left(\frac{6\pi}{13}n + \pi/6\right)}}{2}$$

$$= \frac{1}{2} e^{j\pi/6} \cdot e^{j6\pi/13 n} + \frac{1}{2} e^{-j\pi/6} \cdot e^{-j6\pi/13 n}$$

$$= \frac{1}{2} e^{j\pi/6} \cdot e^{j(3)2\pi/13 n} + \frac{1}{2} e^{-j\pi/6}\, e^{j(-3)2\pi/13 n}$$

$\therefore x[-3] = \frac{1}{2} e^{-j\pi/6}$ $\quad x[3] = \frac{1}{2} e^{j\pi/6}$

Step 3: plot the magnitude and phase plot.

$x[3] = \frac{1}{2}\left[\cos \pi/6 + j \sin \pi/6\right]$ $\quad x[-3] = \frac{1}{2}\left[\cos \pi/6 - j \sin \pi/6\right]$

$= \frac{1}{2}\cos \pi/6 + j \frac{1}{2} \sin \pi/6$ $\quad |x[-3]| = \frac{1}{2}$.

$$|x[3]| = \sqrt{(1/2)^2 \cos^2 \pi/6 + (1/2)^2 \sin^2 \pi/6}$$

$$= (1/2)\sqrt{\cos^2 \pi/6 + \sin^2 \pi/6} = (1/2)$$

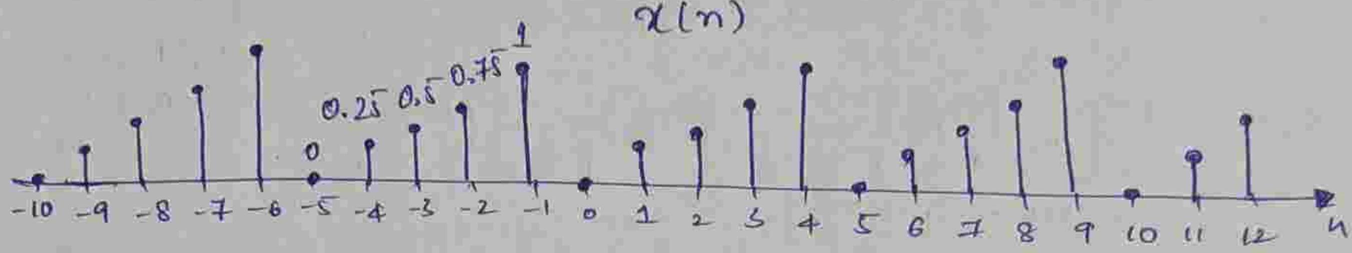
Phase plot:

$X[3] = \frac{1}{2} e^{j\pi/6}$ this is of the form $e^{j\theta}$ $\therefore \angle X[3] = \pi/6$

$X[-3] = \frac{1}{2} e^{-j\pi/6} \Rightarrow \angle X[-3] = -\pi/6 = \tan^{-1}\left[\frac{\sin \pi/6}{\cos \pi/6}\right]$

$= \tan^{-1}[\tan \pi/6] = \pi/6.$

(4) Evaluate the DTFS representation for the signal $x(n)$ shown in the figure and sketch its spectrum.



Step1: find the fundamental period.

By looking at the spectrum the sequence repeats itself after every 5 samples. $\therefore N = 5$.

Angular frequency $\Omega_0 = \frac{2\pi}{N} \cdot m = \frac{2\pi}{5}$

Step2: Represent it interms of sequence of $x(n)$

we know that $X[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}$ — analysis eq^n

$$X[k] = \frac{1}{5}\Big[\cancel{0 \cdot e^{-jk(2\pi/5)0}} + 0.25\, e^{-jk(2\pi/5)1} + 0.5\, e^{-jk(2\pi/5)2} + 0.75\, e^{-jk(2\pi/5)3} + 1 \cdot e^{-jk(2\pi/5)4}\Big]$$

$$= \frac{1}{5}(0.25)\Big[1 \cdot e^{-jk(2\pi/5)1} + 2 \cdot e^{-jk(2\pi/5)2} + 3 e^{-jk(2\pi/5)3} + 4 e^{-jk(2\pi/5)4}\Big]$$

$$\therefore x[k] = \frac{1}{20}\left[e^{-j(2\pi/5)k} + 2e^{-j(4\pi/5)k} + 3.e^{-j(6\pi/5)k} + 4e^{-j(8\pi/5)k}\right] - (1)$$ (12)

Substitute the value of k to obtain the co-efficients.

N = 5 there by k = 0, 1, 2, 3, 4 (one cycle)

$$k=0 \quad x[0] = \frac{1}{20}[1+2+3+4] = 0.5\angle 0^\circ$$

$$x[1] = \frac{1}{20}\left[\underset{\uparrow}{e^{-j(2\pi/5)}} + 2e^{-j4\pi/5} + 3e^{-j(6\pi/5)} + 4e^{-j(8\pi/5)}\right] = 0.21$$

$\cos(2\pi/5) - j\sin(2\pi/5)$

$$= \tfrac{1}{20}\left[0.309 - j0.95 - 1.618 - j1.175 - 2.427 + 1.76j + 1.236 + 3.804 - 2.5 + j3.439\right] = |0.21|$$

$= 0.21\angle 126.02^\circ$

$$x[2] = \frac{1}{20}\left[\underset{\uparrow}{e^{-j(4\pi/5)}} + 2.e^{-j(8\pi/5)} + 3.e^{-j(12\pi/5)} + 4.e^{-j(16\pi/5)}\right]$$

$\cos(144) - j\sin(144)$

$$= \frac{1}{20}\left[-0.809 + 0.5877j + 0.618 + 1902j + 0.9270 - 2.853j - 3.236 + 2.535j\right] = -0.125 + 0.0993 -$$

$0.15\angle 141.5^\circ$

$$x[3] = \frac{1}{20}\left[e^{-j(6\pi/5)} + 2.e^{-j(12\pi/5)} + 3.e^{-j(18\pi/5)} + 4.e^{-j(24\pi/5)}\right]$$

$$= \frac{1}{20}\left[-0.809 + 0.587j + 0.618 - 1.902j + 0.9270 + 2.853j - 3.236 - 2.351j\right] = -0.125 - 0.0406j = 0.13\angle -162^\circ$$

$$x[4] = \frac{1}{20}\left[e^{-j(8\pi/5)} + 2.e^{-j(16\pi/5)} + 3.e^{-j(24\pi/5)} + 4e^{-j(32\pi/5)}\right]$$

$$= \frac{1}{20}\left[0.309 + 0.9510j - 1.618 + 1.175j - 2.427 - 1.763j + 1.236 - 3.804j\right] = -0.125 - 0.1720j = 0.21\angle -126^\circ$$

phase Spectrum and magnitude.

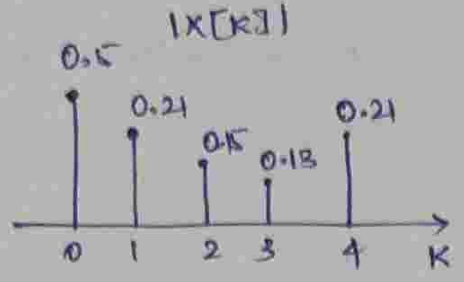
$|x[0]| = 0.5 + 0j = 0.5$ $\quad \angle x[0] = 0$ radians.

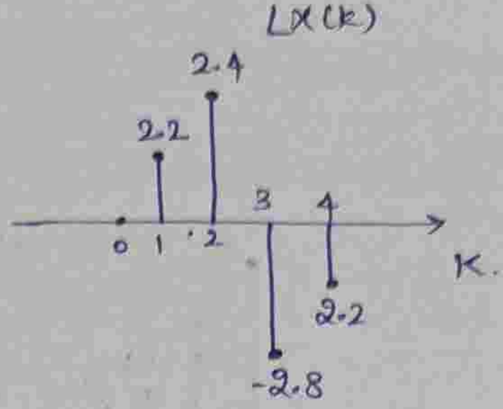
$|x[1]| = -0.125 + 0.1719j = 0.21$ $\quad \angle x[1] = 2.2.$

$|x[2]| = -0.125 + 0.0993j = 0.15$ $\quad \angle x[2] = 2.4$

$|x[3]| = -0.125 + 0.406j = 0.13$ $\quad \angle x[3] = -2.8$

$|X[4]| = -0.125 - 0.1720j = 0.21$ $\quad \angle X[4] = -2.2$ radians.





(5) Consider the Signal.

$$x(n) = 2 + 2\cos\frac{\pi}{4}n + \cos\frac{\pi}{2}n + \frac{1}{2}\cos\frac{3\pi}{4}n.$$

a. Determine and sketch its power density spectrum.

b. Evaluate the power of the signal.

Sol: Angular frequency of 1st term is not present as it is constant.

Angular frequency of 2nd term $= \Omega_{01} = \frac{\pi}{4}$.

Angular frequency of 3rd term $= \Omega_{02} = \frac{\pi}{2}$.

Angular frequency of 4th term $= \Omega_{03} = \frac{\pi}{4}$

$\therefore$ Angular frequency of $x(n) = \gcd(\Omega_{01}, \Omega_{02}, \Omega_{03})$

$$\Omega_0 = \gcd[\pi/4, \pi/2, \pi/4]$$

$$\Omega_0 = \pi/4 = \frac{2\pi}{8} = \frac{2\pi}{N} \quad N = 8 \text{ [fundamental period]}$$

$$\Rightarrow x(n) = 2 + 2\left[\frac{e^{j\pi/4 n} + e^{-j\pi/4 n}}{2}\right] + \left[\frac{e^{j\pi/2 n} + e^{-j\pi/2 n}}{2}\right] + \frac{1}{2}\left[\frac{e^{j3\pi/4} + e^{-j3\pi/4}}{2}\right]$$

$$x(n) = 2 + e^{j(1)\frac{2\pi}{8}n} + e^{j(-1)\frac{2\pi}{8}n} + \frac{1}{2}e^{j(2)(2\pi/8)n} + \frac{1}{2}e^{j(-2)(2\pi/8)n} + \frac{1}{4}e^{j(3)(2\pi/8)n}$$
$$+ \frac{1}{4}e^{j(-3)(2\pi/8)n} \quad - (1)$$

We know that $x(n) = \sum_{k=\langle N\rangle} x[k]\, e^{jk\Omega_0 n} \quad - (2)$ By comparing (1) & (2)

$x[0]=2$. $x[-1]=x[1]=1$. $x[2]=x[-2]=\frac{1}{2}$ $x[3]=x[-3]=\frac{1}{4}$

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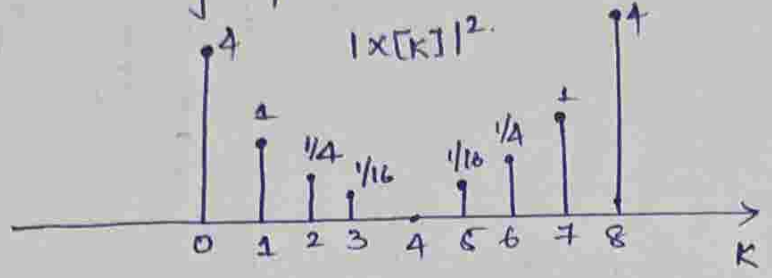
$\therefore x[-1+8]=x[7]=1$ $x[-2+8]=x[6]=\frac{1}{2}$ $x[-3+8]=\frac{1}{4}$ $x[4+8]=0$

Power density spectrum $= |X[k]|^2$



b. The power of the signal [Use Parsavel's theorem]

$$P=\frac{1}{N}\sum_{n=\langle N\rangle}|x(n)|^2=\sum_{K=\langle N\rangle}|X(k)|^2$$

$= 4+1+1+\frac{1}{4}+\frac{1}{4}+\frac{1}{16}+\frac{1}{16}+0$

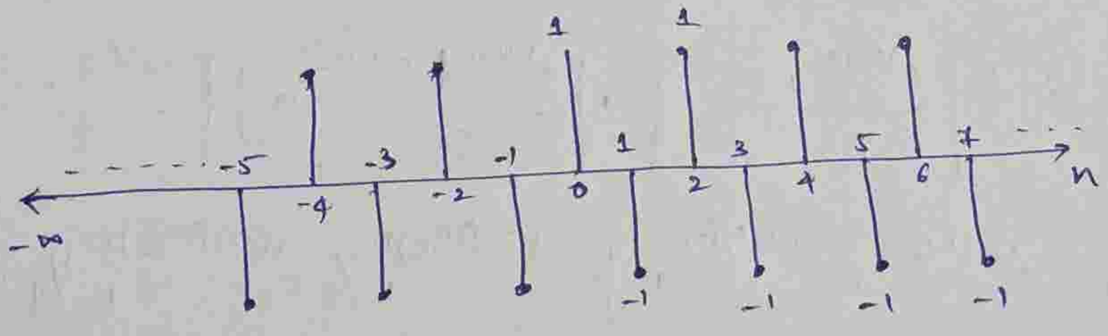
$= X(0)+X[1]+X[7]+X[2]+X[6]+$

$X[5]+X[3]+X[4]$

$= 53/8$.

(6) Determine and sketch the magnitude and phase spectrum of the signal. $x(n)=(-1)^n$; $-\infty<n<\infty$.

Sol: plot the signal.



$\therefore$ fundamental period $= N = 2$.

$$\Omega_0=\frac{2\pi}{N}=\frac{2\pi}{2}=\pi.$$

$$\therefore X[k] = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-jk\Omega_0 n}$$

$$= \frac{1}{2}\sum_{n=0}^{1} x(n)\, e^{-jk\pi n}$$

$$X[k] = \frac{1}{2}\left[x(0)\, e^{-jk\pi.0} + x(1)\, e^{-jk\pi}\right]$$

Write the values from the given graph.

$$X[k] = \frac{1}{2}\left[1 - 1\, e^{-jk\pi}\right]$$

$$\therefore X[0] = \tfrac{1}{2}[1-1] = 0+0j$$

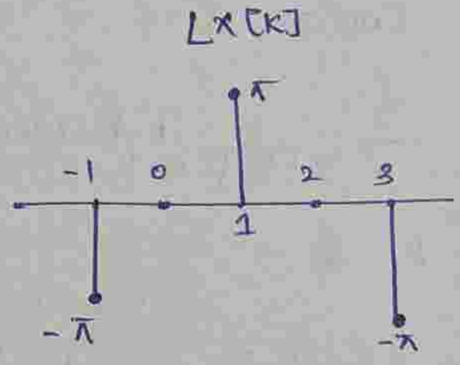
$$X[1] = \tfrac{1}{2}[1-e^{-j\pi}] = \tfrac{1}{2}[1-\cos\pi + j\sin\pi] = \tfrac{1}{2}[1-\cos 180^\circ + j\sin 180^\circ]$$

$$= \tfrac{1}{2}[1+1+0] = 1+0j$$

$\therefore$ magnitude spectrum and phase spectrum.

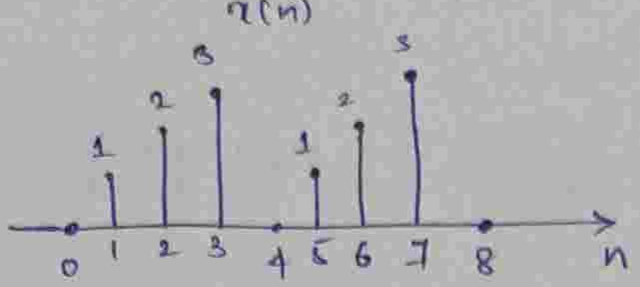
$X[0] = 0+0j \qquad |X[0]| = 0 \qquad \angle X[0] = 0$ radians.

$X[1] = 1+0j \qquad |X[1]| = 1 \qquad \angle X[1] = \pi$ radians.



7) ~~Determine DTFS representation for the signal $x(n) = \cos(n\pi/3)$ plot the spectrum of $X[k]$~~

(7) Find the DTFS Co-efficients of the signals shown (16)



Co-efficients of DTFS are given by

$$x[k] = \frac{1}{N} \sum_{n=\langle N \rangle} x(n)\, e^{-jk\Omega_0 n}$$

By looking at the spectrum the sequence repeats itself after every 4 samples $\therefore N = 4$.

Angular frequency $\Omega_0 = \frac{2\pi}{N} \cdot m = \frac{2\pi}{4} = \frac{\pi}{2}$.

$$\Rightarrow x[k] = \frac{1}{4} \sum_{n=0}^{3} x(n)\, e^{-jk(\pi/2)n}.$$

$$x[k] = \frac{1}{4}\Big[x(0) \cdot e^{-jk(\pi/2)\cdot 0} + x(1)\, e^{-jk(\pi/2)\cdot 1} + x(2) \cdot e^{-jk(\pi/2)\cdot 2} + x(3) \cdot e^{-jk(\pi/2)\cdot 3}\Big]$$

$$= \frac{1}{4}\Big[1 e^{-jk\pi/2} + 2 \cdot e^{-jk\pi} + 3\, e^{-jk3\pi/2}\Big]$$

$$= \frac{1}{4}\Big[(e^{-j\pi/2})^k + 2 \cdot (e^{-j\pi})^k + 3 \cdot (e^{-j3\pi/2})^k\Big]$$

$$e^{-j\pi/2} = \cos\pi/2 - j\sin\pi/2 = -j \qquad e^{-j3\pi/2} = \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$e^{-j\pi} = \cos\pi - j\sin\pi = -1 \qquad = +j$$

$$x(k) = \frac{1}{4}\left[(-j)^k + 2(-1)^k + 3(j)^k\right]$$

$$x(0) = \frac{1}{4}\left[(-j)^0 + 2(-1)^0 + 3(j)^0\right] = \frac{1}{4}\left[1+2+3\right] = \frac{6}{4} = \frac{3}{2}.$$

$$x[1] = \frac{1}{4}\left[(-j)^1 + 2(-1)^1 + 3(j)^1\right] = \frac{1}{4}\left[-j-2+3j\right] = \frac{-2+2j}{4}$$

$$x[2] = \frac{1}{4}\left[(-j)^2 + 2(-1)^2 + 3(j)^2\right] = \frac{1}{4}\left[-1+2-3\right] = -2/4 = -1/2$$

$$x[3] = \frac{1}{4}\left[(-j)^3 + 2(-1)^3 + 3(+j)^3\right] = \frac{1}{4}\left[+j-2-3j\right] = \frac{-2-2j}{4}$$

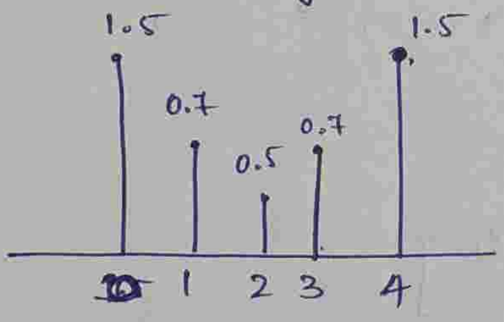
magnitude and phase spectrum.

$x(0) = 3/2 + 0j \Rightarrow |x(0)| = 3/2 = 1.5 \quad \angle x(0) = 0.$

$x(1) = -1/2 + 1/2 j \Rightarrow |x(0)| = 0.70 \quad \angle x(1) = 2.3$

$x(2) = -1/2 + 0j \Rightarrow |x(2)| = 1/2 = 0.5 \quad \angle x(2) = 3.14$

$x(3) = -1/2 - 1/2 j \Rightarrow |x(3)| = 0.70 \quad \angle x(3) = -2.3$



* Continuous - time periodic Signals - Continuous time fourier Series [CTFS or FS]

frequency domain representation of time domain Signal.

A Continuous time fourier series representation of periodic Signal $x(t)$ is given by.

$$x(t) = \sum_{K=-\infty}^{\infty} X[K]\, e^{jK\omega_0 t} \quad -(1) \text{ Synthesis equation}$$

$$X[K] = \frac{1}{T}\int_{T} x(t)\, e^{-jK\omega_0 t}\, dt. \quad -(2) \text{ Analysis equation}$$

where $x(t)$ has a fundamental period T

and fundamental frequency $\omega_0 = \frac{2\pi}{T}$ (radians/sec)

$X[K]$ = fourier series coefficients of $x(n)$

- when i/p signal is Continuous and periodic then $x(n)$ and $X(K)$ form a fourier Series pair denoted by

$$x(t) \xleftrightarrow{FS;\ \omega_0} X[K]$$

time domain representation

frequency domain Representation

- Magnitude spectrum of $x(t)$ is given by $|X(K)|$
- Phase Spectrum of $x(t)$ is given by $\angle X(K)$.

* Properties of Continuous time Fourier Series :-

1. **Linearity** :- If $x(t) \xleftrightarrow{FS;\omega_0} X[k]$ and $y(t) \xleftrightarrow{FS;\omega_0} Y[k]$ then

$$z(t) = ax(t) + by(t) \xleftrightarrow{FS;\omega_0} Z[k] = aX[k] + bY[k]$$

In this case both $x(t)$ and $y(t)$ have the same fundamental period $T = \frac{2\pi}{\omega_0}$.

Proof:
$$X[k] = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$Y[k] = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$\therefore Z(k) = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T [ax(t) + by(t)]\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}\int_T ax(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}\int_T by(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T}a\int_T x(t)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T}b\int_T y(t)\, e^{-jk\omega_0 t}\, dt$$

$$Z[k] = aX[k] + bY[k]$$

hence proved.

(2) Time Shift:-

If $x(t) \xleftrightarrow{FS;\omega_0} X[k]$ then.

$$y(t) = x(t-t_0) \xleftrightarrow{FS;\omega_0} Y[k] = e^{-jk\omega_0 t_0} X[k]$$

Proof:- $X[k] = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$

$$\therefore Y[k] = \frac{1}{T}\int_T y(t) \cdot e^{-jk\omega_0 t} dt$$

$$= \frac{1}{T}\int_T x(t-t_0)\, e^{-jk\omega_0 t} \cdot dt$$

Put $t - t_0 = m$.

$$Y[k] = \frac{1}{T}\int_T x(m) \cdot e^{-jk\omega_0(m+t_0)} dt$$

$$= e^{-jk\omega_0 t_0} \cdot \underbrace{\frac{1}{T}\int_T x(m) \cdot e^{-jk\omega_0 m} dt}_{\text{Analysis equation.}}$$

$$= e^{-jk\omega_0 t_0} \cdot X[k]$$

Hence proved.

(3) Frequency Shift:- If $x(t) \xleftrightarrow{FS;\omega_0} X[k]$

then $y(t) = e^{jk_0\omega_0 t} \cdot x(t)$; $y(t) \xleftrightarrow{FS;\omega_0} Y[k] = X[k-k_0]$

Proof:- $X[k] = \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t} dt$

$$\therefore Y[k] = \frac{1}{T}\int_T y(t)\, e^{-jk\omega_0 t} dt$$

$$Y[K] = \frac{1}{T} \int_T e^{jK_0 \omega_0 t} \cdot x(t) \cdot e^{-jK\omega_0 t}$$

$$= \frac{1}{T} \int x(t) \cdot e^{-j(K-K_0)\omega_0 t} \, dt$$: comparing it with standard formule.

$$= X[K-K_0]$$

④ <u>Scaling</u>: if $x(t) \xleftrightarrow{FS;\,\omega_0} X[K]$ then.

$$Z(t) = x(at) \xleftrightarrow{FS;\,\omega_0} Z[K] = X(K)$$

Proof: $$Z(t) = \frac{1}{T} \int_T Z(t) \cdot e^{-jK\omega_0 t} \, dt$$

If $x(t)$ is periodic then $Z(t) = x(at)$ is also periodic. ∴ fundamental period $T = T/a$.

$$\Rightarrow Z(K) = \frac{1}{T/a} \int_{\langle T/a \rangle} Z(t) \cdot e^{-jK(a\omega_0)t} \, dt$$

$$= \frac{1}{T/a} \int_{\langle T/a \rangle} x(at) \, e^{-jK(a\omega_0)t} \, dt$$

Substitute $at = m$ then $dt = \frac{1}{a} dm$.

$$\therefore Z(K) = \frac{a}{T} \int_{\langle T/a \rangle} x(m) \cdot e^{-jK\omega_0 m} \cdot \frac{1}{a} \, dm$$

$$= \frac{1}{T} \int_{\langle T/a \rangle} x(m) \, e^{-jK\omega_0 m} \, dm.$$

$$= X[K]$$ proved.

(5) Time-differentiation :- If $x(t) \xleftrightarrow{FS;\,\omega_0} x[k]$ then

$$\frac{dx(t)}{dt} \xleftrightarrow{FS;\,\omega_0} jk\omega_0\, x[k]$$

Proof :- We know that

$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{jk\omega_0 t}$$

differentiating both the side w.r.t time t we get

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\sum_{k=-\infty}^{\infty} x[k]\, e^{jk\omega_0 t}\right]$$

Change the order of differentiation and Σ.

$$\Rightarrow \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} x[k] \cdot \frac{d}{dt} e^{jk\omega_0 t}$$

$$\frac{dx(t)}{dt} = \left[\sum_{k=-\infty}^{\infty} x[k] \cdot e^{jk\omega_0 t}\right] jk\omega_0$$

$$\therefore \frac{dx(t)}{dt} = jk\omega_0 \cdot x[k]$$

hence proved.

(6) Convolution :- If $x(t) \xleftrightarrow{FS;\,\omega_0} x[k]$ and $y(t) \xleftrightarrow{FS;\,\omega_0} Y[k]$; $\omega_0 = \frac{2\pi}{T}$ then.

$$z(t) = x(t) \circledast y(t) \xleftrightarrow{FS;\,\omega_0} Z[k] = T \cdot x[k] \cdot Y[k]$$

$\circledast \rightarrow$ periodic Convolution.

Proof: $X[k] = \frac{1}{T} \int_T x(t)\, e^{-jk\omega_0 t}\, dt$

$$Y[k] = \frac{1}{T} \int_T y(t) \cdot e^{-jk\omega_0 t}\, dt$$

$$\therefore Z[k] = \frac{1}{T} \int_T z(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \frac{1}{T} \int_T [x(t) \circledast y(t)]\, e^{-jk\omega_0 t}\, dt$$

Using the defination of periodic convolution, we get

$$Z[k] = \frac{1}{T} \int_{t<T>} \left[\int_{l<T>} x(l) \cdot y(t-l)\, dl \right] e^{-jk\omega_0 t}\, dt$$

Change the order of integration, we get.

$$Z[k] = \frac{1}{T} \int_{l=T} x(l) \int_{t=T} y(t-l)\, e^{-jk\omega_0 t}\, dl \cdot dt$$

Put $t - l = m$ $\therefore dt = dm$, then

$$Z[k] = \frac{1}{T} \left[\int_{l=T} x(l) \int_{m=T} y(m) \cdot e^{-jk\omega_0 (m+l)}\, dl \cdot dm \right]$$

$$= \frac{1}{T} \left[\int_{l=T} x(l) \cdot e^{-jk\omega_0 l}\, dl \cdot \int_{m=T} y(m)\, e^{-jk\omega_0 m}\, dm \right]$$

$$= \frac{1}{\cancel{T}} \left[T \cdot X[k] \times \cancel{T}\, Y[k] \right]$$

$Z[k] = T \cdot [X[k] \cdot Y[k]]$ hence proved.

(7) Modulation :- If $x(t) \xleftrightarrow{FS;\omega_0} X[k]$ and $y(t) \xleftrightarrow{FS;\omega_0} Y[k]$

then $Z(t) = x(t).y(t) \xleftrightarrow{FS;\omega_0} Z[k] = X[k] * Y[k]$

Proof : We know that $Z[k] = \frac{1}{T}\int_T z(t)\, e^{-jk\omega_0 t}\, dt$.

$$Z[k] = \frac{1}{T}\int_T x(t).y(t)\, e^{-jk\omega_0 t}\, dt \quad -(1)$$

we know that $x(t) = \sum_{l=-\infty}^{\infty} X[l]\, e^{jl\omega_0 t}$

Substituting in equation (1) we have.

$$Z[k] = \frac{1}{T}\int_T \left[\sum_{l=-\infty}^{\infty} X[l]\, e^{j\omega_0 l t}\right] y(t)\, e^{-jk\omega_0 t}\, dt$$

changing the order of summation and integration,

$$Z[k] = \frac{1}{T}\left[\sum_{l=-\infty}^{\infty} X[l] \int_T y(t)\, e^{-j(k-l)\omega_0 t}\, dt\right]$$

$$= \cancel{\frac{1}{T}}\left[\sum_{l=-\infty}^{\infty} X[l] \cdot Y[k-l]\right]$$ By defination of convolution.

$$Z[k] = X[k] * Y[k]$$

hence proved.

(8) Parseval's theorem :- If $x(t) \xleftrightarrow{FS;\omega_0} X[k]$ then

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |X[k]|^2$$

Proof:- LHS of the above equation is the average power of a periodic continuous-time signal $x(t)$ with fundamental period T. i.e $P = \frac{1}{T}\int_{T} |x(t)|^2 dt$

this equation can we written as.

$$P = \frac{1}{T}\int_{T} x(t)\, x^*(t)\, dt$$

$$P = \frac{1}{T}\int_{T} x(t)\left[\sum_{k=-\infty}^{\infty} x^*[k]\, e^{-jk\omega_0 t}\right] dt$$

Changing the order of summation and integration,

$$P = \frac{1}{T}\sum_{k=-\infty}^{\infty} x^*[k] \underbrace{\int_{T} x(t)\, e^{-jk\omega_0 t}\, dt}_{\text{use defination}}$$

$$= \cancel{\frac{T}{T}}\sum_{k=-\infty}^{\infty} x^*[k]\cdot x[k]$$

$$= \sum_{k=-\infty}^{\infty} |x[k]|^2$$

$$\therefore \frac{1}{T}\int_{T} |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |x[k]|^2.$$

the Sequence $|x[k]|^2$ for $k = 0, 1, 2 \ldots$ is the distribution of power as a function of frequency and it is called "<u>power density spectrum</u>" of the Signal $x(t)$

9) Symmetry :- If $x(t) \xleftrightarrow{FS;\, \omega_0} X[k]$ then

$x(t)$ real $\xleftrightarrow{FS;\, \omega_0}$ $X^*[k] = X[-k]$

$x(t)$ img $\xleftrightarrow{FS;\, \omega_0}$ $X^*[k] = -X[-k]$

Problems :-

1) For the signal $x(t) = \sin \omega_0 t$; find the fourier Series and draw its spectrum.

Sol: given $x(t) = \sin \omega_0 t$

$$x(t) = \frac{e^{j\omega_0 t} - e^{-j\omega_0 t}}{2j} = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t} \quad -(1)$$

We know that $x(t) = \sum_{k=-\infty}^{\infty} X[k]\, e^{jk\omega_0 t}$

Represent eq (1) interms of $x(t)$ to obtain $X[k]$ Sequence

$$x(t) = \frac{1}{2j} e^{j(1)\omega_0 t} - \frac{1}{2j} e^{j(-1)\omega_0 t}$$

$\therefore X[1] = 1/2j \qquad X[-1] = -1/2j$

magnitude Spectrum : ~~$|X[k]| = |1/2j|$~~ j/j

$X[1] = \frac{1}{+2} j = +0.5j = 0 - 0.5j = \sqrt{0^2 + (0.5)^2} = 0.5 = 1/2$

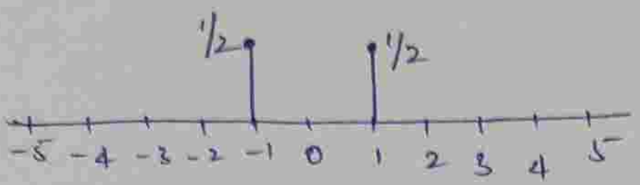
$X[-1] = \frac{-1}{+2} j = -0.5j = 0 + 0.5j = \sqrt{0^2 + (0.5)^2} = 0.5 = 1/2$

Phase spectrum : $\angle X[k] = \tan^{-1}(b/a)$

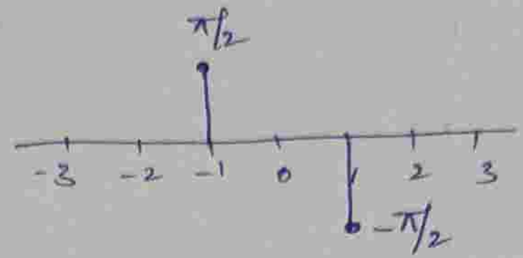
$\angle X[1] = \tan^{-1}(0.5/0) = \tan^{-1}(\infty) = +\pi/2$

$\angle X[-1] = \tan^{-1}(0.5/0) = \tan^{-1}(\infty) = -\pi/2$

Magnitude Spectrum $|X[K]|$



Phase Spectrum.



2) Evaluate the FS representation for the signal. $x(t) = \sin(2\pi t) + \cos(3\pi t)$. Sketch magnitude and phase spectrum.

given :- $x(t) = \sin 2\pi t + \cos 3\pi t$

1st term angular frequency $\omega_{0_1} = 2\pi$

2nd term angular frequency $\omega_{0_2} = 3\pi$

$\therefore$ angular frequency $\omega_0 = \gcd(\omega_{0_1}, \omega_{0_2}) = \gcd(2\pi, 3\pi)$

$$\boxed{\omega_0 = \pi}$$

We know that $x(t) = \sum_{K=-\infty}^{\infty} X[K] \, e^{jK\omega_0 n}$ — ①

$$x(t) = \sin 2\pi t + \cos 3\pi t$$

$$= \frac{1}{2j} e^{j2\pi t} - \frac{1}{2j} e^{-j2\pi t} + \frac{1}{2} e^{j3\pi t} + \frac{1}{2} e^{-j3\pi t}$$

$$= \frac{1}{2j} e^{j(2)\pi t} - \frac{1}{2j} e^{j(-2)\pi t} + \frac{1}{2} e^{j(3)\pi t} + \frac{1}{2} e^{+j(-3)\pi t} \quad \text{—②}$$

Comparing equation ① and ②

$X[2] = 0 + 0.5j$ $\qquad$ $X[3] = 0.5 + 0j$

$X[-2] = 0 - 0.5j$ $\qquad$ $X[-3] = 0.5 + 0j$

magnitude and phase spectrum :

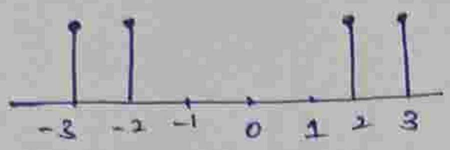
$|x(2)| = 0.5 \qquad \angle x(2) = \pi/2$

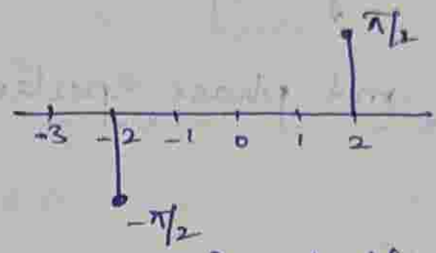
$|x(-2)| = 0.5 \qquad \angle x(-2) = -\pi/2$

$|x(3)| = 0.5 \qquad \angle x(3) = 0$

$|x(-3)| = 0.5 \qquad \angle x(-3) = 0$

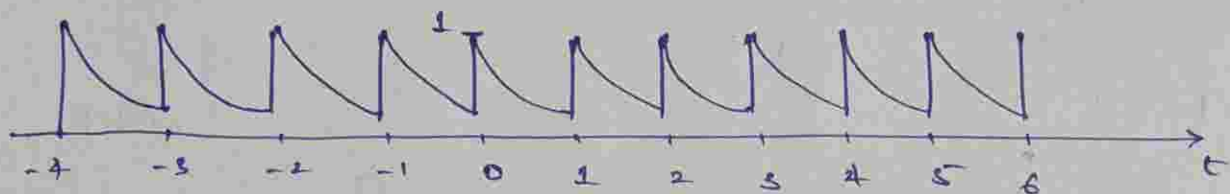
$|x(k)|$





(3) For the Signal $x(t)$ shown below, find the FS representation and draw its magnitude and phase spectra.

$x(t)$



Sol: from the given signal $x(t) = e^{-t}$ [decaying signal]

Signal is periodic and $T = 1$.

$$W_0 = \frac{2\pi}{T} = 2\pi$$

We know that $x[k] = \frac{1}{T} \int_{t=\langle T \rangle} x(t)\, e^{-jkW_0 t} \cdot dt$

$$\therefore x[k] = \frac{1}{1} \int_{t=0}^{1} e^{-t} \cdot e^{-jkW_0 t} \cdot dt$$

$$= \int_{t=0}^{1} e^{-(1+jkW_0)t} \cdot dt = -\frac{e^{-(1+jkW_0)t}}{(1+jkW_0)} \Bigg|_0^1$$

$\omega_0 = 2\pi$

$$= -\left.\frac{e^{-(1+j2\pi k)t}}{(1+j2\pi k)}\right|_0^1 = \frac{1-e^{-1}}{1+j2\pi k}.$$

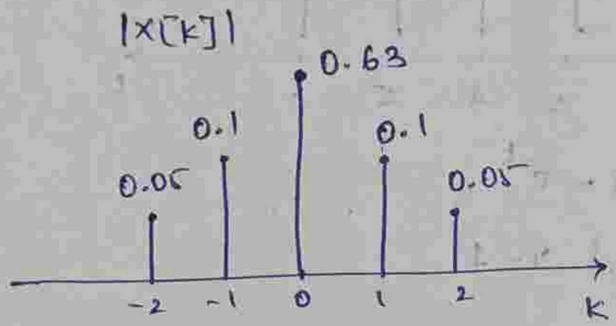
$$\boxed{x[k] = \frac{1-e^{-1}}{1+2\pi jk}.}$$

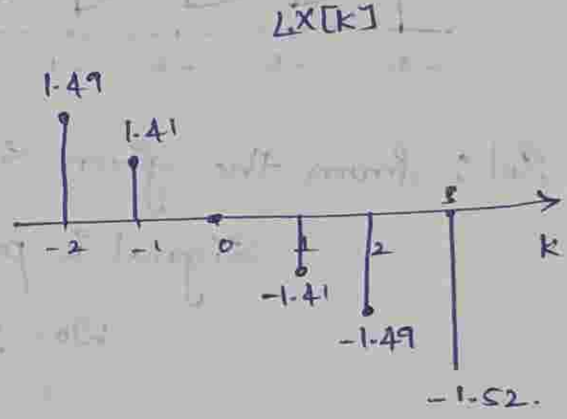
Magnitude and phase spectrum :-

$x[0] = \frac{1-e^{-1}}{1+0} = 1-e^{-1} = 0.63+0j \Rightarrow |x[0]| = 0.63 \quad \angle X[0] = 0°$

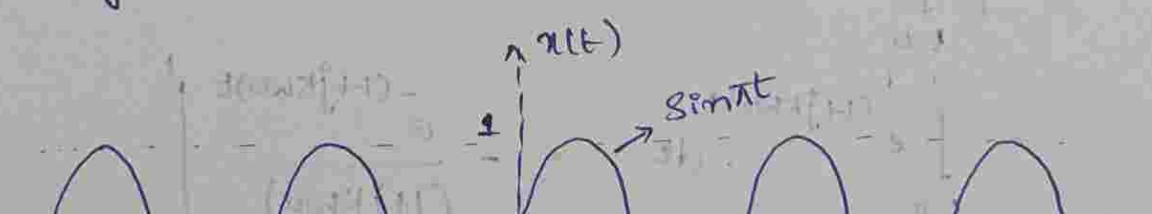
$x[1] = \frac{1-e^{-1}}{1+2\pi j} = \frac{1-e^{-1}}{1+6.28j} = 0.015-0.097j \Rightarrow |x[1]| = 0.1 \quad \angle X[1] = -1.41$

$x[2] = \frac{1-e^{-1}}{1+4\pi j} = \frac{1-e^{-1}}{1+12.56j} = 0.0039-0.0498j \Rightarrow |x[2]| = 0.05 \quad \angle X[2] = -1.49$





(4) Find the FS Coefficient for the signal $x(t)$ shown in the figure below.



from the figure the sample repeats after two intervals

$\therefore T = 2 \quad \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2} = \pi$

We know that $x[k] = \frac{1}{T} \int_{t=\langle T \rangle} x(t)\, e^{-jk\omega_0 t}\, dt$

$$x[k] = \frac{1}{2} \int_0^1 x(t) . e^{-jk\pi t}\, dt$$

$$x[k] = \frac{1}{2} \int_0^1 \sin \pi t . e^{-jk\pi t}\, dt$$

$$= \frac{1}{2} \int_0^1 \left[\frac{e^{j\pi t} - e^{-j\pi t}}{2j} \right] e^{-jk\pi t}\, dt$$

$$= \frac{1}{4j} \left[\int_0^1 e^{j(1-k)\pi t}\, dt - \int_0^1 e^{-j(1+k)\pi t}\, dt \right]$$

$$= \frac{1}{4j} \left[\frac{e^{j(1-k)\pi t}}{j(1-k)\pi} \Bigg|_0^1 - \frac{e^{-j(1+k)\pi t}}{-j(1+k)\pi} \Bigg|_0^1 \right]$$

$e^{-j\pi} = e^{j\pi} = (-1)$

$$= \frac{1}{4j} \left[\frac{e^{j(1-k)\pi} - e^{0}}{j(1-k)\pi} + \frac{e^{-j(1+k)\pi} - e^{0}}{j(1+k)\pi} \right]$$

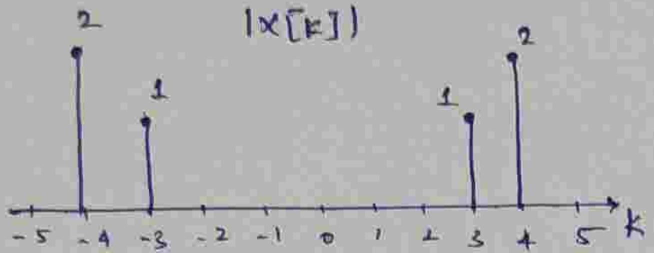
$$= \frac{1}{4j} \left[\frac{(-1)^{(1+k)} - 1}{j(1-k)\pi} + \frac{(-1)^{(1+k)} - 1}{j(1+k)\pi} \right]$$

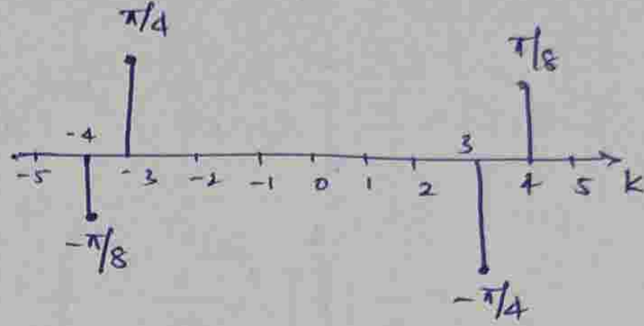
$$= \frac{1}{4j^2\pi} \left[(-1)^{(k+1)} - 1 \right] \left[\frac{1}{(1-k)} + \frac{1}{(1+k)} \right]$$

$$= -\frac{1}{\cancel{4}\pi} \left[(-1)^{k+1} - 1 \right] \left[\frac{\cancel{2}}{1-k^2} \right]$$

$$x[k] = \frac{1}{2\pi}\left[\frac{1-(-1)^{k+1}}{1-k^2}\right]$$

(5) Determine the time signal corresponding to the magnitude and phase spectra shown in the figure with $w_0 = \pi$.





from the figure given.

$x[4] = 2 \cdot e^{+j\pi/8}$ $\qquad$ $x[-4] = 2\, e^{-j\pi/8}$

$x[3] = 1 \cdot e^{-j\pi/4}$ $\qquad$ $x[3] = 1 \cdot e^{+j\pi/4}$

we know that $x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{jkw_0 t}$

$$x(t) = x[-4]\, e^{j(-4)\pi t} + x[-3]\, e^{j(-3)\pi t} + x[3]\, e^{j(3)\pi t} + x[4]\, e^{j4\pi t}$$

$$= 2\, e^{-j\pi/8} \cdot e^{j(-4)\pi t} + e^{j\pi/4} \cdot e^{j(-3)\pi t} + e^{-j\pi/4} \cdot e^{j3\pi t} + 2 e^{j\pi/8} \cdot e^{j4\pi t}$$

$$= \left[e^{j(3\pi t - \pi/4)} + e^{-j(3\pi t - \pi/4)}\right] + 2\left[e^{j(4\pi t + \pi/8)} + e^{-j(4\pi t + \pi/8)}\right]$$

$$= 2\cos(3\pi t - \pi/4) + 4\cos(4\pi t + \pi/8)$$

⑥ Determine the FS representation for the signal.

$x(t) = \cos 4t + \sin 8t.$

Angular frequency of 1st term = 4

Angular frequency of 2nd term = 8

$\therefore$ Angular frequency of $x(t)$ is gcd of $(4, 8) = 4$.

$\therefore W_0 = 4$.

$$x(t) = \sum_{k=-\infty}^{\infty} x[k]\, e^{jk w_0 t} \quad \text{--- ①}$$

$$\Rightarrow x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-j4t} + \frac{1}{2j} e^{j8t} - \frac{1}{2j} e^{-j8t}$$

$$= \frac{1}{2} e^{j(1)4t} + \frac{1}{2} e^{j(-1)4t} + \frac{1}{2j} e^{j(2)4t} - \frac{1}{2j} e^{j(-2)4t} \quad \text{--- ②}$$

Comparing eq ① and eq ② we have.

$x[1] = 1/2 \qquad x[-1] = 1/2$

$x[2] = 1/2j \qquad x[-2] = -1/2j$

$x[k] = 0$ for $k \neq \pm 1, \pm 2$.

* Discrete time non periodic signals

1. Discrete time fourier transforms (DTFT).

the discrete time fourier series representation of a periodic sequence $x(n)$ is given by

$$x(n) = \frac{1}{2\pi} \int_{2\pi} X(e^{j\Omega}) \cdot e^{j\Omega n} \cdot d\Omega$$

or it can also be written as.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\Omega}) \cdot e^{j\Omega n}\, d\Omega \quad -(1)$$ Synthesis equation

where $$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} \quad -(2)$$ Analysis equation

$X(e^{j\Omega})$ is known as Discrete-time fourier transform (DTFT) of the signal $x(n)$. the relationship b/w $X(e^{j\Omega})$ and $x(n)$ can be expressed as.

$$x(n) \xleftrightarrow{DTFT} X(e^{j\Omega}).$$

Here $X(e^{j\Omega}) \rightarrow$ frequency domain representation.

$x(n) \rightarrow$ time domain signal.

$X(e^{j\Omega})$ is also know as spectrum of $x(n)$

Let us consider,

$$X(e^{j(\Omega+2k\pi)}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\Omega+2k\pi)n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n} \cdot \cancel{e^{-j2k\pi n}}.$$ 1 irrespective of k & n values.

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}$$

$$\therefore X(e^{j(\Omega+2k\pi)}) = X(e^{j\Omega})$$

this indicates $X(e^{j\Omega})$ is periodic with period 2π.

* Properties of DTFT :-

1. **Linearity** :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ and $y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$

then $z(n) = ax(n) + by(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = aX(e^{j\Omega}) + bY(e^{j\Omega})$.

Proof :- We know that $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\Omega n} \quad \therefore \quad Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n) \cdot e^{-j\Omega n}$$

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} [ax(n) + by(n)]\, e^{-j\Omega n}$$

$$= a \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n} + b \cdot \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\Omega n}$$

$$= a\, X(e^{j\Omega}) + b\, Y(e^{j\Omega})$$

Hence proved.

2. **Time Shift** :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ then

$$p(n) = x(n-n_0) \longleftrightarrow P(e^{j\Omega}) = e^{-j\Omega n_0} \cdot X(e^{j\Omega}).$$

Proof : We have, $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}$

$$\therefore \; P(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} p(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} x(n-n_0) \cdot e^{-j\Omega n}$$

Put $n - n_0 = m$, then

$$P(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\Omega(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot e^{-j\Omega m} \cdot e^{-j\Omega n_0}$$

$$P(e^{j\Omega}) = e^{-j\Omega n_0} \cdot X(e^{j\Omega})$$

Hence proved.

3. Frequency shift :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ then.

$$y(n) = e^{j\beta n} x(n) \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j(\Omega-\beta)})$$

Proof: We know that by defination

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\Omega n}.$$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\Omega n}.$$

$$= \sum_{n=-\infty}^{\infty} e^{j\beta n} \cdot x(n) \cdot e^{-j\Omega n}.$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j(\Omega-\beta)n}$$

$$Y(e^{j\Omega}) = X(e^{j(\Omega-\beta)})$$

Hence the proof.

4. Scaling :-

Scaling of a discrete-time signal discards informations it is not possible to express DTFT of a Scaled Signal in, terms of DTFT of the original Signal.

But Considering a non-periodic Sequence $x(n)$, such that

$x(n) = 0$; unless $\frac{n}{p}$ is integer. ; $p > 1$.

then $z[n] = x(pn)$ is also non-peridic.

In this case, if

$$x(n) \xleftrightarrow{DTFT} X(e^{j\Omega}) \text{ then.}$$

$$z(n) = x(pn) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = X(e^{j\Omega/p})$$

By defination $Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n) \cdot e^{-j\Omega n}$.

$$= \sum_{n=-\infty}^{\infty} x(pn) \cdot e^{-j\Omega n}.$$

Put $pn = m$, then.

$$Z(e^{j\Omega}) = \sum_{m=-\infty}^{\infty} x(m) \times e^{-j(\Omega/p) \cdot m}$$

$$Z(e^{j\Omega}) = X(e^{j(\Omega/p)})$$

Hence proved.

(5) Frequency - differentiation:-

If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ then.

$$-jn\, x(n) \longleftrightarrow \frac{d}{d\Omega} X(e^{j\Omega})$$

Proof: We have by defination

$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}. \quad -(1)$$

Differentiating both the sides wrt Ω, we get

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \frac{d}{d\Omega}\left[\sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}\right]$$

Changing the order of differentiation and summation, we get.

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\left[\frac{d}{d\Omega} e^{-j\Omega n}\right]$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot (-jn)\, e^{-j\Omega n}.$$

$$\frac{d}{d\Omega} X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} (-jn\, x(n))\, e^{-j\Omega n}. \quad -(2)$$

By comparing equation (2) with equation (1) we get,

$$-jn x(n) \xleftrightarrow{DTFT} \frac{d}{d\Omega} X(e^{j\Omega}).$$

(6) Summation:- If $x(n) \longleftrightarrow X(e^{j\Omega})$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \longleftrightarrow Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{1-e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$$

Proof: We know that summation is the reverse process of differencing. the summation operation on $x(n)$ yields $y(n)$ where as the difference operation on $y(n)$ will yield $x(n)$.

i.e $x(n) = y(n) - y(n-1)$

Taking DTFT on both the sides, we get

$$X(e^{j\Omega}) = Y(e^{j\Omega}) - e^{-j\Omega} Y(e^{j\Omega})$$ → Using time shift property

$$Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} \quad \text{— (i)}$$

From above equation, we cannot determine $Y(e^{j0})$. therefore we add an impulse to account for a non zero average value in $x(k)$, to get the relationship as.

$$Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} + \pi X(e^{j0})\,\delta(\Omega) \quad ; -\pi < \Omega < \pi$$

the first term of the impulse series is assumed to be zero for $\Omega = 0$. $Y(e^{j\Omega})$ is periodic 2π, hence we can write it as.

$$Y(e^{j\Omega}) = \frac{X(e^{j\Omega})}{1 - e^{-j\Omega}} + \pi X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\Omega - k2\pi)$$

(7) Convolution:- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ and $y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$ then $z(n) = x(n) * y(n)$

$$z(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = X(e^{j\Omega}) \cdot Y(e^{j\Omega})$$

Proof:- We have, $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n)\, e^{-j\Omega n}$$

$$\therefore Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) * y(n)] \, e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} \left[\sum_{l=-\infty}^{\infty} x(l) \cdot y(n-l) \right] e^{-j\Omega n}$$

Changing the order of summations, we get

$$Z(e^{j\Omega}) = \sum_{l=-\infty}^{\infty} x(l) \sum_{n=-\infty}^{\infty} y(n-l) \, e^{-j\Omega n}$$

Put $n - l = m$.

$$Z(e^{j\Omega}) = \sum_{l=-\infty}^{\infty} x(l) \sum_{m=-\infty}^{\infty} y(m) \cdot e^{-j\Omega(m+l)}$$

$$= \sum_{l=-\infty}^{\infty} x(l) \cdot e^{-j\Omega l} \cdot \sum_{m=-\infty}^{\infty} y(m) \, e^{-j\Omega m}$$

$$\therefore Z(e^{j\Omega}) = X(e^{j\Omega}) \cdot Y(e^{j\Omega})$$

therefore Convolution in time domain is equivalent to multiplication in frequency domain.

(8) Modulation :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ and $y(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$ then $z(n) = x(n) \cdot y(n)$

$$z(n) \xleftrightarrow{DTFT} Z(e^{j\Omega}) = \frac{1}{2\pi} [X(e^{j\Omega}) \circledast Y(e^{j\Omega})]$$

periodic Convolution

Proof :- We know that $X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}$

$$Y(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j\Omega n}$$

$$\therefore Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} z(n) \cdot e^{-j\Omega n}$$

$$= \sum_{n=-\infty}^{\infty} [x(n) \cdot y(n)] \, e^{-j\Omega n}$$

By defination $x(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\beta}) \, e^{j\beta n} \, d\beta$ in the above eq

$$Z(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} y(n) \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\beta}) \, e^{j\beta n} \, d\beta \right] e^{-j\Omega n}$$

Interchanging the order of integration and summation, we get

$$Z(e^{j\Omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\beta}) \sum_{n=-\infty}^{\infty} y(n) \cdot e^{j\beta n} \cdot e^{-j\Omega n} \, d\beta.$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\beta}) \sum_{n=-\infty}^{\infty} y(n) \cdot e^{-j(\Omega-\beta)n} \, d\beta.$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\beta}) \cdot Y(e^{j(\Omega-\beta)n}) \, d\beta$$ → Represents the convolution of $X(e^{j\Omega})$ and $Y(e^{j\Omega})$

$$= \frac{1}{2\pi} \left[X(e^{j\Omega}) \circledast Y(e^{j\Omega}) \right]$$

hence proved.

(9) Parseval's theorem:- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ then.

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 \, d\Omega.$$

$|X(e^{j\Omega})|^2$ is known as energy density spectrum of signal $x(n)$

LHS of the above equation is the energy of signal $x(n)$.

Proof:- $$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$= \sum_{n=-\infty}^{\infty} x(n) \cdot x^*(n)$$

$$= \sum_{n=-\infty}^{\infty} x(n) \left[\frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega}) \cdot e^{-j\Omega n} \, d\Omega \right]$$

Changing the order of summation and integration

$$E = \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega}) \underbrace{\sum_{n=-\infty}^{\infty} x(n) \cdot e^{-j\Omega n}}_{} \, d\Omega.$$

By defination.

$$= \frac{1}{2\pi} \int_{2\pi} X^*(e^{j\Omega}) \cdot X(e^{j\Omega}) \, d\Omega.$$

$$E = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 \, d\Omega.$$

$$\therefore \sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi} \int_{2\pi} |X(e^{j\Omega})|^2 \, d\Omega.$$

(10) Symmetry :- If $x(n) \xleftrightarrow{DTFT} X(e^{j\Omega})$ then

$x(n)$ real $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = X(e^{-j\Omega})$

$x(n)$ imaginary $\xleftrightarrow{DTFT} X^*(e^{j\Omega}) = -X(e^{-j\Omega})$

$x(n)$ real and even $\longleftrightarrow \text{Img}\{X(e^{j\Omega})\} = 0$

$x(n)$ real and odd $\longleftrightarrow \text{Re}\{X(e^{j\Omega})\} = 0$.

Problems:-

(1) Find the DTFT for the following signal $x(n)$ and draw its spectrum. $x(n) = a^n u(n)$; $|a| < 1$

Sol: From defination of DTFT we have

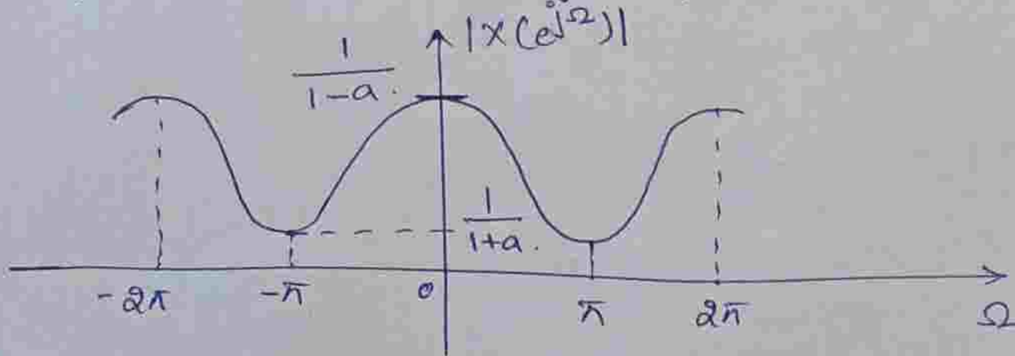
$$X(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} a^n . u(n) . e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (a e^{-j\Omega})^n$$

$$X(e^{j\Omega}) = \frac{1}{1 - a e^{-j\Omega}} = \frac{e^{+j\Omega}}{e^{j\Omega} - a}.$$

Substituting values for Ω we get spectrum as



To obtain magnitude and phase spectrum

$$X(e^{j\Omega}) = X(\Omega) = \frac{1}{1-ae^{-j\Omega}} = \frac{1}{1-a[\cos\Omega - j\sin\Omega]}$$

$$= \frac{1}{(1-a\cos\Omega)+ja\sin\Omega}$$

$$= \frac{1}{(1-a\cos\Omega)+ja\sin\Omega} \times \frac{(1-a\cos\Omega)-ja\sin\Omega}{(1-a\cos\Omega)-ja\sin\Omega}$$

$$= \frac{(1-a\cos\Omega)-ja\sin\Omega}{(1-a\cos\Omega)^2+(a\sin\Omega)^2}$$

$$= \frac{(1-a\cos\Omega)-ja\sin\Omega}{1-2a\cos\Omega+a^2\cos^2\Omega+a^2\sin^2\Omega}$$

$$= \frac{(1-a\cos\Omega)-ja\sin\Omega}{1-2a\cos\Omega+a^2}$$

$$= \frac{1-a\cos\Omega}{1-2a\cos\Omega+a^2} - j\frac{a\sin\Omega}{1-2a\cos\Omega+a^2}$$

$$|X(\Omega)| = \frac{1}{\sqrt{1-2a\cos\Omega+a^2}}$$

$$\angle X(\Omega) = \tan^{-1}\left[\frac{-a\sin\Omega}{1-a\cos\Omega}\right]$$

(2) Find DTFT of the signal $x(n) = (-1)^n u(n)$

Sol: From defination we have $X(\Omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\Omega n}$.

$$\therefore X(\Omega) = \sum_{n=0}^{\infty} (-1)^n \cdot e^{-j\Omega n}$$

$$= \sum_{n=0}^{\infty} (-e^{-j\Omega})^n = \frac{1}{1+e^{-j\Omega}}$$

$$X(\Omega) = \frac{1}{e^{-j\Omega/2} \cdot e^{j\Omega/2} + e^{-j\Omega/2} \cdot e^{-j\Omega/2}}$$

$$= \frac{1}{e^{-j\Omega/2}\left[e^{j\Omega/2} + e^{-j\Omega/2}\right]} = \frac{1}{e^{-j\Omega/2} \cdot 2\cos(\Omega/2)}$$

$$X(\Omega) = \frac{e^{j\Omega/2}}{2\cos(\Omega/2)}$$

(3) Determine the discrete time Sequence where DTFT is given as

$$X(\Omega) = 1 \quad \text{for } -\Omega_c \leq \Omega \leq \Omega_c$$

$$= 0 \quad \text{for } \Omega_c \leq |\Omega| \leq \pi$$

Sol: the inverse DTFT is given by defination.

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega n}\, d\Omega$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\Omega n}}{jn}\right]_{-\Omega_c}^{\Omega_c}$$

$$= \frac{1}{2\pi}\left[\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{jn}\right]$$

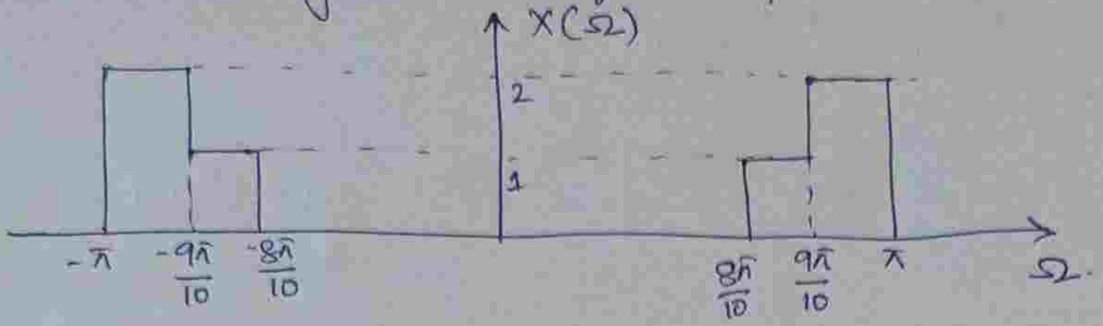
$$= \frac{1}{n\pi}\left[\frac{e^{j\Omega_c n} - e^{-j\Omega_c n}}{2j}\right] = \frac{1}{n\pi}\sin(\Omega_c n) \; ; \; n \neq 0$$

with $n = 0$.

$$x(0) = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1 \cdot e^{j\Omega(0)}\, d\Omega = \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} 1\, d\Omega.$$

$$= \frac{1}{2\pi}\left[\Omega\right]_{-\Omega_c}^{\Omega_c} = \frac{\Omega_c}{\pi} \; ; \; n = 0.$$

(4) Determine the signal $x(n)$ if its spectrum is as shown.



Sol: from defination we have

$$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\Omega)\, e^{j\Omega n}\, d\Omega$$

$$x(n) = \frac{1}{2\pi}\left[\int_{-\pi}^{-9\pi/8} 2 \cdot e^{j\Omega n}\, d\Omega + \int_{-9\pi/10}^{-8\pi/10} 1 \cdot e^{j\Omega n}\, d\Omega + \int_{8\pi/10}^{9\pi/10} 1 \cdot e^{j\Omega n}\, d\Omega + \int_{9\pi/10}^{\pi} 2 \cdot e^{j\Omega n}\, d\Omega\right]$$

$$= \frac{1}{2\pi}\left[2 \cdot \frac{e^{j\Omega n}}{jn}\Bigg|_{-\pi}^{-9\pi/10} + \frac{e^{j\Omega n}}{jn}\Bigg|_{-9\pi/10}^{-8\pi/10} + \frac{e^{j\Omega n}}{jn}\Bigg|_{8\pi/10}^{9\pi/10} + 2 \cdot \frac{e^{j\Omega n}}{jn}\Bigg|_{9\pi/10}^{\pi}\right]$$

$$x(n) = \frac{1}{n\pi}\left[\sin n\pi - \sin\left(\frac{8\pi n}{10}\right) - \sin\left(\frac{9\pi n}{10}\right)\right]$$

(5) Find the inverse DTFT for $X(\Omega) = \dfrac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$ using appropriate properties.

Sol:- given. $$X(\Omega) = \frac{6}{e^{-j2\Omega} - 5e^{-j\Omega} + 6}$$

$$= \frac{6}{(e^{-j\Omega} - 2)(e^{-j\Omega} - 3)}$$

using partial fraction we can write it as

$$\frac{6}{(e^{-j\Omega} - 2)(e^{-j\Omega} - 3)} = \frac{A}{(e^{-j\Omega} - 2)} + \frac{B}{(e^{-j\Omega} - 3)}$$

Solve for A and B values.

$$x(\Omega) = \frac{-6}{(e^{-j\Omega}-2)} + \frac{6}{(e^{-j\Omega}-3)}$$

$$= \frac{-6}{-2(1-\frac{1}{2}e^{-j\Omega})} + \frac{6}{-3(1-\frac{1}{3}e^{-j\Omega})}$$

$$= \frac{3}{(1-\frac{1}{2}e^{-j\Omega})} + \frac{(-2)}{(1-\frac{1}{3}e^{-j\Omega})}$$

We know that

$$a^n u(n) \xleftrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}$$

$$\therefore \quad x(n) = \left[3(\tfrac{1}{2})^n u(n) - 2(\tfrac{1}{3})^n u(n)\right]$$

$$= \left[3(\tfrac{1}{2})^n - 2(\tfrac{1}{3})^n\right] u(n).$$

(6) Using the appropriate property, find DTFT of the following Signal. $x(n) = (\frac{1}{2})^n u(n-2)$.

<u>Sol</u>:- given $x(n) = (\frac{1}{2})^n u(n-2)$

we can rewrite the given as.

$$x(n) = (\tfrac{1}{2})^2 (\tfrac{1}{2})^{n-2} u(n-2)$$

We know that $a^n u(n) \xrightarrow{DTFT} \frac{1}{1-ae^{-j\Omega}}$.

$$\therefore (\tfrac{1}{2})^n u(n) \xrightarrow{DTFT} \frac{1}{1-\frac{1}{2}e^{-j\Omega}}.$$

$\therefore$ Using time shift property.

$$x(n-n_0) \xrightarrow{DTFT} e^{-j\Omega n_0} x(\Omega)$$

$$(\tfrac{1}{2})^{n-2} u(n-2) \longrightarrow e^{-j2\Omega} \cdot \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

Using linearity property

$$\left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^{n-2} u(n-2) \longrightarrow \left(\tfrac{1}{2}\right)^2 e^{-j2\Omega} \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

$$\Rightarrow \quad X(\Omega) = \frac{1}{4} e^{-j2\Omega} \cdot \frac{1}{1-\frac{1}{2}e^{-j\Omega}}$$

* **Continuous time non-periodic signal : Fourier Transform [CTFT / FT]**

A non-periodic continuous-time signal $x(t)$ can be expressed as,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} \, d\omega \rightarrow \text{Synthesis eq}^n$$

where $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} \, dt \rightarrow$ analysis eq^n

$X(\omega)$ is known as fourier transform of $x(t)$

$X(\omega)$ and $x(t)$ forms a FT pair, which can be expressed as. $x(t) \xleftrightarrow{FT} X(\omega)$

$X(\omega) \rightarrow$ frequency domain representation

$x(t) \rightarrow$ time domain representation

* **Properties of Fourier transform:-**

(1) **Linearity:** If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \longleftrightarrow Y(\omega)$

then $z(t) = ax(t) + by(t) \xleftrightarrow{FT} Z(\omega) = aX(\omega) + bY(\omega)$

Proof:- $Z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$

$$= \int_{-\infty}^{\infty} [a x(t) + b y(t)]\, e^{-j\omega t}\, dt.$$

$$= a\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} + b \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}$$

using analysis equation

$$Z(\omega) = aX(\omega) + bY(\omega)$$

Hence the proof.

② Time Shift:

If $x(t) \xleftrightarrow{FT} X(\omega)$ and $y(t) \xleftrightarrow{FT} Y(\omega)$ then.

$$y(t) = x(t-t_0) \xleftrightarrow{FT} Y(\omega) = e^{-j\omega t_0} X(\omega).$$

Proof: From defination we have.

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt.$$

Put $t - t_0 = m$. then $dt = dm$.

$$Y(\omega) = \int_{-\infty}^{\infty} x(m)\, e^{-j\omega(m+t_0)}\, . dm.$$

$$= \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m} . e^{-j\omega t_0}\, dm.$$

$$= e^{-j\omega t_0} \int_{-\infty}^{\infty} x(m)\, e^{-j\omega m}\, dm.$$

$$= e^{-j\omega t_0} . X(\omega)$$

③ Frequency Shift:-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$y(t) = e^{j\beta t}\, x(t) \xleftrightarrow{FT} Y(\omega) = X(\omega-\beta)$$

Proof:- From defination we have.

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(t) \cdot e^{j\beta t} \cdot e^{-j\omega t}\, dt.$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega-\beta)t}\, dt.$$

$$Y(\omega) = X(\omega-\beta)$$

Hence the proof.

④ Scaling:

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$y(t) = x(at) \xleftrightarrow{FT} Y(\omega) = \frac{1}{|a|} X(\omega/a)$$

Proof: From defination we have.

$$Y(\omega) = \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, dt.$$

$$Y(\omega) = \int_{-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt.$$

Case 1: If a is positive i.e $a > 0$

$at = \tau$

$dt = 1/a\, d\tau$

$\therefore Y(\omega) = \frac{1}{a} \int_{-\infty}^{\infty} x(\tau)\, e^{-j(\omega/a)\tau} . d\tau$

from defination we can write it as.

$Y(\omega) = \frac{1}{a} X(\omega/a)$ — ①

Case 2 :- If a is negative i.e $a < 0$

Put $-at = \tau$

$dt = -1/a\, d\tau$

$Y(\omega) = -\frac{1}{a} \int_{-\infty}^{\infty} x(-\tau)\, e^{-j(\omega/a)(-\tau)} d\tau$

$Y(\omega) = -\frac{1}{a} X(\omega/a)$ — ②

$\therefore$ from equation ① and ② we get.

$Y(\omega) = \frac{1}{|a|} X(\omega/a)$.

Hence the proof.

⑤ Time differentiation :-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$\frac{dx(t)}{dt} \longleftrightarrow j\omega X(\omega)$.

Proof : from defination we have.

$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega$. — ①

differentiating both the sides with respect to t we get.

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega\right]$$

changing the order of differentiation and integration

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\left[\frac{d}{dt}\left(e^{j\omega t}\right)\right] d\omega.$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, j\omega \,.\, e^{j\omega t}\, d\omega$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left[j\omega\, X(\omega)\right] e^{j\omega t}\, d\omega \quad -(2)$$

Comparing equation (2) with equation (1) we get.

$$\frac{dx(t)}{dt} \xleftrightarrow{FT} j\omega\, X(\omega).$$

Hence the proof.

(6) Frequency Differentiation:-

If $x(t) \xleftrightarrow{FT} X(\omega)$

then $-jt\, x(t) \xleftrightarrow{FT} \frac{d}{d\omega} X(\omega)$

Proof: We know that.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \quad -(1)$$

differentiating both the sides w.r.t. ω, we get

$$\frac{dX(\omega)}{d\omega} = \frac{d}{d\omega}\left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt\right]$$

Interchanging the Order of differentiation and integration, we get

$$\frac{dX(\omega)}{dt} = \int_{-\infty}^{\infty} x(t) \cdot \left[\frac{d}{d\omega} e^{-j\omega t}\right] dt.$$

$$= \int_{-\infty}^{\infty} x(t) \left(-j t \, e^{-j\omega t}\right) dt.$$

$$\frac{dX(\omega)}{dt} = \int_{-\infty}^{\infty} \left[-jt \, x(t)\right] e^{-j\omega t} dt \quad - (2)$$

Comparing with equation (2) and (1) we get.

$$-jt \, x(t) \xleftrightarrow{FT} \frac{dX(\omega)}{d\omega}.$$

hence the proof.

(7) Integration:

If $x(t) \longleftrightarrow X(\omega)$ then.

$$\int_{-\infty}^{t} x(\tau) d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(0) \delta(\omega)$$

Proof:- We know that

$$x_1(t) * x_2(t) = \int_{-\infty}^{\infty} x_1(\tau) \, x_2(t-\tau) \, d\tau$$

Let $x_1(t) = x(t)$ and $x_2(t) = u(t)$

$$\therefore x(t) * u(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot u(t-\tau) \, d\tau. \quad - (1)$$

we know that $u(t-\tau) = 1 \quad ; t-\tau \geq 0$

$= 0 \quad ; t-\tau < 0.$

there fore eq ① reduces / rewritten as.

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, u(t-\tau)\, d\tau + \int_{t}^{+\infty} x(\tau)\, u(t-\tau)\, d\tau$$

$\therefore$ we can write the above equation as.

$$x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$$

Taking fourier transform on both the sides.

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = F\left[x(t) * u(t)\right]$$

Using Convolution property. we get.

$$F\left[\int_{-\infty}^{t} x(\tau)\, d\tau\right] = X(\omega)\,.\, FT[u(\omega)]$$

$$= X(\omega) \cdot \left[\pi\delta(\omega) + \frac{1}{j\omega}\right]$$

$$= \frac{X(\omega)}{j\omega} + X(\omega)\,\pi\,\delta(\omega)$$

$$\therefore \int_{-\infty}^{t} x(\tau)\, d\tau \xleftrightarrow{FT} \frac{X(\omega)}{j\omega} + \pi X(0)\, \delta(\omega)$$

We assume $X(0)$ to be equal to 1. which will help us to prove the property.

3) Convolution :-

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$z(t) = x(t) * y(t) \xleftrightarrow{FT} Z(\omega) = X(\omega)\,.\,Y(\omega)$$

Proof: From the defination we have.

$$Z(w) = \int_{-\infty}^{\infty} z(t)\, e^{-jwt}\, dt$$

$$= \int_{-\infty}^{\infty} [x(t) * y(t)]\, e^{-jwt}\, dt$$

By using the defination of Convolution we get

$$= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \right] e^{-jwt}\, dt.$$

$$= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} y(t-\tau)\, e^{-jwt}\, dt \right] d\tau.$$

Put $t - \tau = m$ and $dt = dm$.

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} y(m)\, e^{-jw(m+\tau)}\, dm \,.\, d\tau.$$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-jw\tau}\, d\tau \,.\, \int_{-\infty}^{\infty} y(m)\, e^{-jwm}\, dm.$$

$$Z(w) = X(w) \,.\, Y(w)$$

convolution in time domain is equivalent to multiplication in frequency domain.

(9) Modulation:-

If $x(t) \xleftrightarrow{FT} X(w)$ and $y(t) \xleftrightarrow{FT} Y(w)$ then,

$$z(t) = x(t) \cdot y(t) \xleftrightarrow{FT} Z(w) = \frac{1}{2\pi} [X(w) * Y(w)]$$

Proof:- From defination we have.

$$X(w) = \int_{-\infty}^{\infty} x(t)\, e^{-jwt}\, dw$$

$$z(\omega) = \int_{-\infty}^{\infty} z(t)\, e^{-j\omega t}\, d\omega.$$

$$z(\omega) = \int_{-\infty}^{\infty} [x(t) \cdot y(t)]\, e^{-j\omega t}\, d\omega.$$

We know that $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\beta)\, e^{j\beta t}\, d\beta.$

$$\therefore z(\omega) = \int_{-\infty}^{\infty} \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X(\beta)\, e^{j\beta t}\, d\beta\right] y(t)\, e^{-j\omega t}\, dt.$$

Changing the Order of integration, we get

$$z(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\beta) \int_{-\infty}^{\infty} y(t)\, e^{-j\omega t}\, e^{j\beta t}\, dt\, d\beta$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\beta) \int_{-\infty}^{\infty} y(t)\, e^{-j(\omega-\beta)t}\, dt\, d\beta.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\beta)\, Y(\omega-\beta)\, d\beta.$$

$$= \frac{1}{2\pi} [X(\beta) * Y(\beta)]$$

multiplication in time domain is equivalent to Convolution in frequency domain.

(10) <u>Parsavel's theorem:-</u>

If $x(t) \xleftrightarrow{FT} X(\omega)$ then.

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega$$

Proof: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\, dt$

$$= \int_{-\infty}^{\infty} x(t) \left[\frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega)\, e^{-j\omega t} d\omega \right] dt.$$

Changing the Order of integration, we get.

$$E = \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \left[\int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \right] d\omega$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} X^*(\omega) \cdot X(\omega)\, d\omega.$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

$$\therefore \int_{-\infty}^{\infty} |x(t)|^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2 d\omega$$

Hence the proof.

Problems:-

1. Obtain the Fourier. Transform of the Signal.

$$x(t) = e^{-at} u(t) \; ; \; a > 0.$$

Draw its magnitude and phase spectrum.

Sol: from defination we have.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt.$$

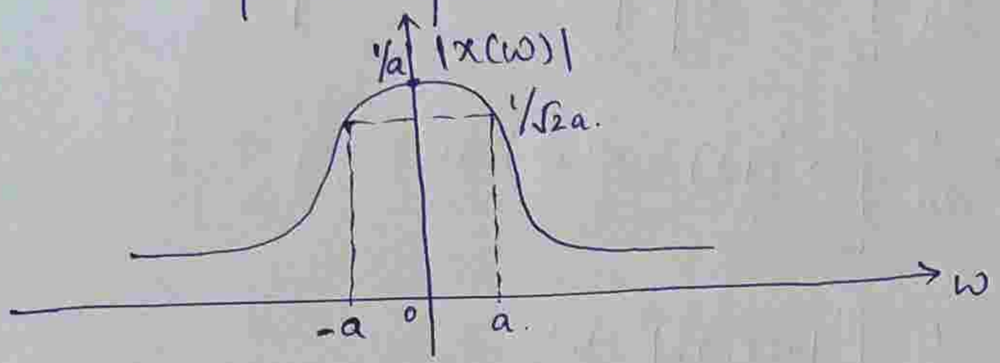
$$X(\omega) = \int_{0}^{\infty} e^{-at} \cdot e^{-j\omega t} dt$$

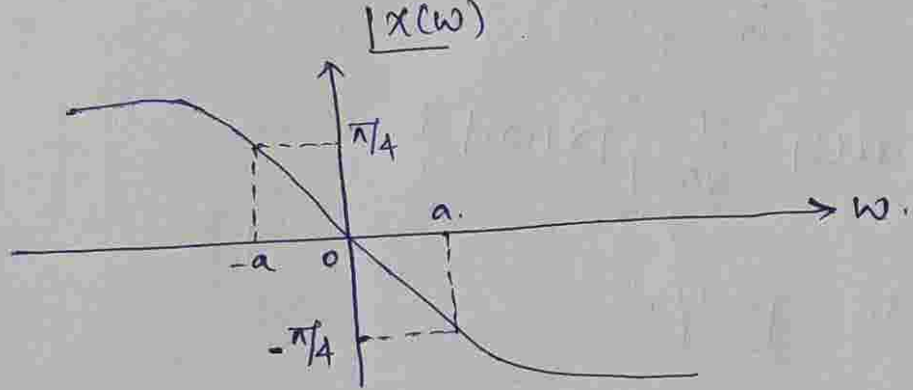
$$X(\omega) = \int_0^{\infty} e^{-(a+j\omega)t}\, dt$$

$$= \left. \frac{e^{-(a+j\omega)t}}{-(a+j\omega)} \right|_0^{\infty} = \frac{1}{a+j\omega}$$

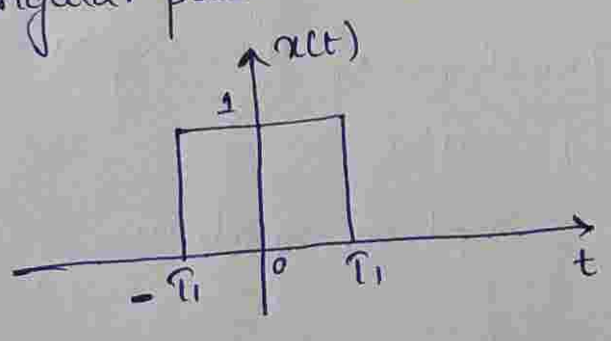
$$|X(\omega)| = \frac{1}{a^2+\omega^2} \quad ; \quad \angle X(\omega) = -\tan^{-1}(\omega/a)$$

the magnitude and phase spectra are as shown below.





(2) For the rectangular pulse shown in fig, draw the spectrum.



Sol:- from the fig $x(t) = 1 \quad ; -T_1 \le t \le T_1$

$\qquad\qquad 0 \quad ;$ otherwise.

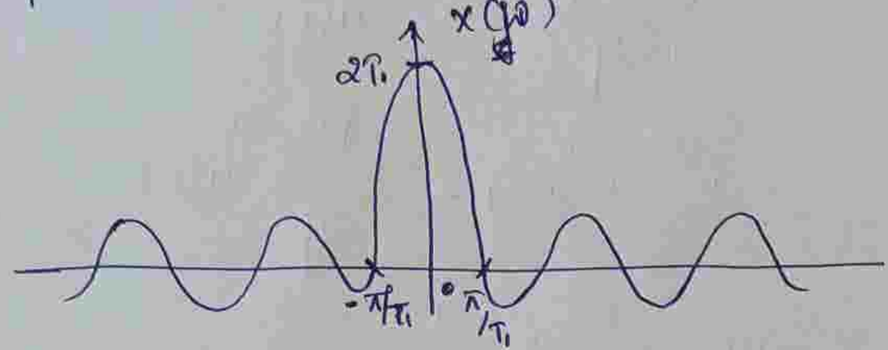
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$X(\omega) = \int_{-T_1}^{T_1} 1 \cdot e^{-j\omega t}\, dt = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T_1}^{T_1}$$

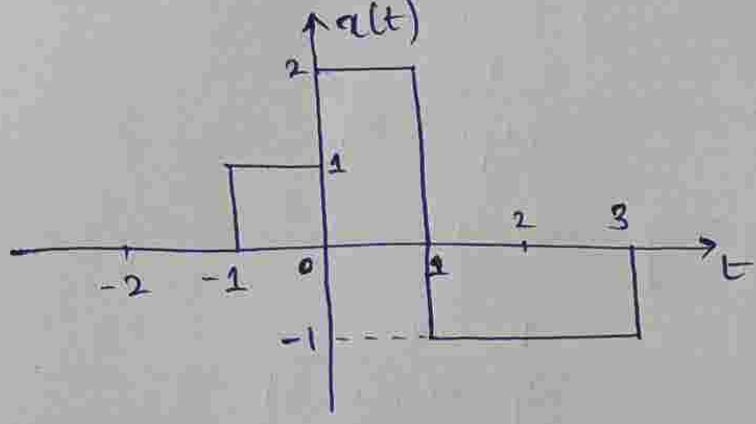
$$= \frac{1}{-j\omega}\left[e^{-j\omega T_1} - e^{+j\omega T_1} \right]$$

$$= \frac{2\sin \omega T_1}{\omega}.$$

Spectrum ploted as.



(8) Compute the Fourier transform for the Signal $x(t)$



Sol:- From the figure.

$$x(t) = 1 \quad ; -1 < t < 0.$$
$$= 2 \quad ; 0 < t < 1.$$
$$= -1 \quad ; 1 < t < 3.$$
$$= 0 \quad ; \text{otherwise}.$$

we know that $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$

$$= \int_{-1}^{0} 1 \cdot e^{-j\omega t}\, dt + \int_{0}^{1} 2\, e^{-j\omega t}\, dt - \int_{1}^{3} 1 e^{-j\omega t}\, dt$$

$$x(\omega) = \frac{1}{j\omega}\left[1 + e^{j\omega} - 2e^{-j\omega} + e^{-j3\omega}\right]$$

4. Find the Fourier transform of the following Signal using appropriate properties.

$$x(t) = \sin \pi t \cdot e^{-2t} u(t)$$

Sol:- given $x(t) = \sin \pi t \, e^{-2t} u(t)$

$$= \frac{e^{j\pi t} - e^{-j\pi t}}{2j} \cdot e^{-2t} \cdot u(t)$$

$$= \frac{e^{j\pi t} \cdot e^{-2t}}{2j} u(t) - \frac{e^{-j\pi t} e^{-2t}}{2j} u(t)$$

We know that $e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{2+j\omega}$

Using frequency shifting property, we get.

$$e^{j\pi t} e^{-2t} u(t) \longleftrightarrow \frac{1}{2+j(\omega-\pi)}$$

Using linearity property, we get.

$$\frac{1}{2j} e^{j\pi t} e^{-2t} u(t) \longleftrightarrow \frac{1}{2j} \cdot \frac{1}{2+j(\omega-\pi)}$$

$$\therefore x(t) = \sin \pi t \, e^{-2t} u(t) \xleftrightarrow{FT} \frac{1}{2j}\left[\frac{1}{2+j(\omega-\pi)} - \frac{1}{2+j(\omega-\pi)}\right]$$

5. Compute the Fourier transform of the Signal.

$$x(t) = 1+\cos \pi t \quad ; |t| \leq 1.$$

$$= 0 \quad ; |t| > 1.$$