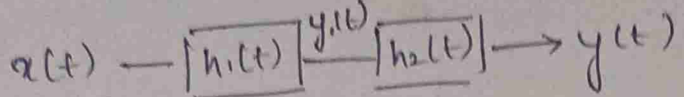
LTI System has inter connection when Connected in (1)
parallel and series.

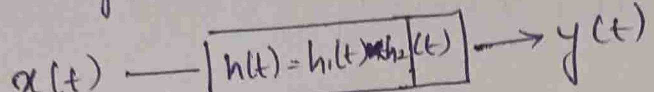
Unit 3
Representation of LTI System

1. Cascade Connection of system :-

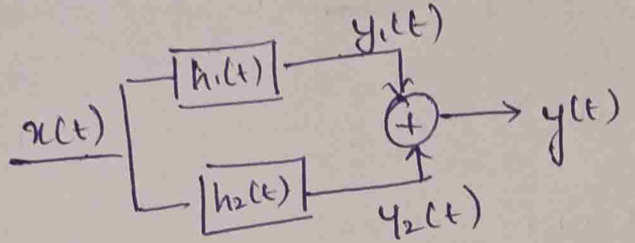


using this we can obtain commutative & associative properties of system.

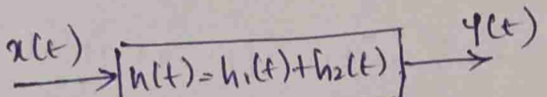
∴ this system canbe represented as.



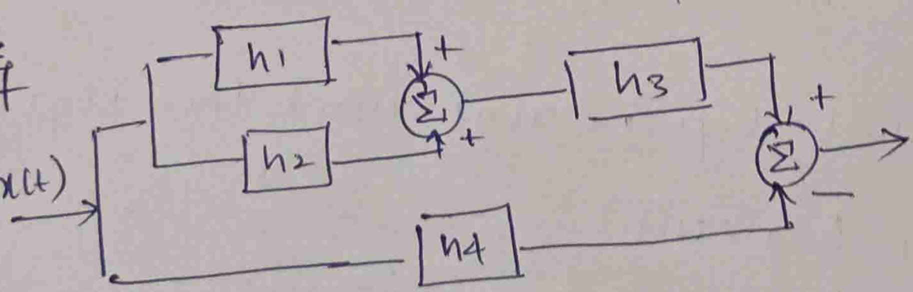
2. Parallel Connections of system.
this is used to obtain distributive law.



$$y(t) = x(t) * h_1(t) + x(t) * h_2(t)$$
$$= y_1(t) + y_2(t)$$

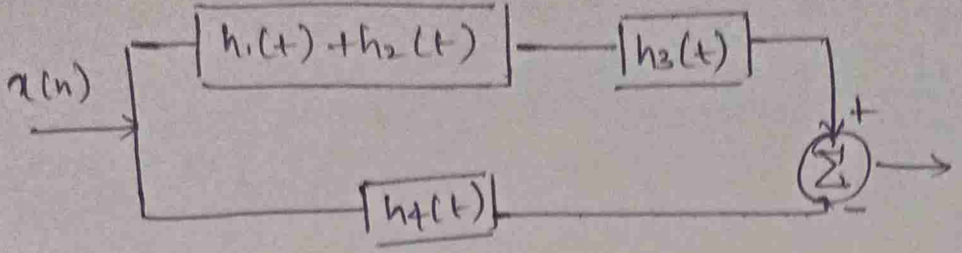


Eg

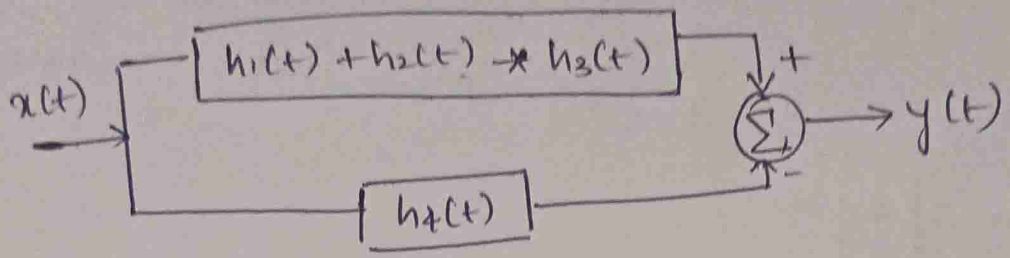


$h_1(t) = u(n)$ $h_2(t) = u(n+2) - u(n)$

$h_3(t) = \delta(n-2)$ $h_4 = a^n u(n)$



⇒



$$\therefore\ y(t) = [h_1(t) + h_2(t) * h_3(t)] - h_4(t)$$

$$[[\cancel{u(n)} + u(n+2) - \cancel{u(n)}] * \delta(n-2)] - a^n u(n)$$

$$= [u(n+2) * \delta(n-2)] - a^n u(n)$$

$$= u(n) - a^n u(n)$$

$$= (1 - a^n)\, u(n).$$

* Unit Step Response of a LTI System.

The o/p $y(n)$ of a discrete-time LTI system characterized by an impulse response $h(n)$ with i/p $x(n)$ is.

$$y(n) = h(n) * x(n)$$

If i/p is a unit step i.e $x(n) = u(n)$ then step response

$$s(n) = h(n) * u(n)$$

$$= \sum_{k=-\infty}^{\infty} h(k).\, u(n-k)$$

We know that $u(n-k) = 1 \quad ; n-k \geq 0 \quad k \leq n$
$\qquad\qquad\qquad\quad = 0 \quad ; n-k < 0 \quad k > n.$

$$\therefore s(n) = \sum_{k=-\infty}^{n} h(k)$$

$\therefore$ Step responce of a LTI system is the running sum of the impulse response.

Similarly. $y(t) = h(t) * x(t)$

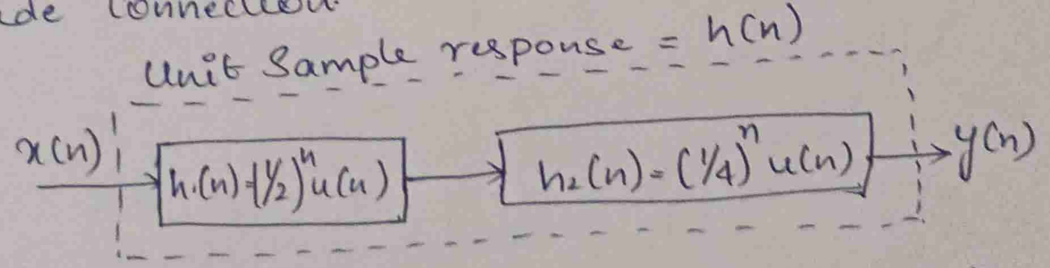
If $x(t) = u(t)$, then Step response $y(t) = s(t)$ is

$$\therefore s(t) = h(t) * u(t)$$
$$= \int_{-\infty}^{\infty} h(\tau) \cdot u(t-\tau)\, d\tau$$

We know that $u(t-\tau) = 1 \quad ; t-\tau \geq 0$ or $\tau \leq t$
$\qquad\qquad\qquad\quad = 0 \quad ; t-\tau < 0$ or $\tau > t$

$$\therefore \boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau.}$$

Q. Two discrete time LTI Systems are connected in cascade as shown. Determine the unit sample response of this cascade connection



Cascade Connection $\therefore h(n) = h_1(n) * h_2(n)$

By Convolution defination:

$$h(n) = \sum_{k=-\infty}^{\infty} h_1(k) \cdot h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (1/2)^k u(k) \cdot (1/4)^{n-k} u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} (1/2)^k u(k) (1/4)^n (1/4)^{-k} u(n-k)$$

$$= (1/4)^n \sum_{k=-\infty}^{\infty} (1/2)^k (1/4)^{-k} u(k) . u(n-k)$$

Using defination $u(k)\,u(n-k) = \begin{cases} 1 ; n \geq k \\ 0 ; n < k \end{cases}$

$$h(n) = (1/4)^n \sum_{k=+0}^{n} (1/2)^k (1/4)^{-k}.$$

$$= (1/4)^n \sum_{k=+0}^{n} (1/2)^k (4)^k.$$

$$= (1/4)^n \sum_{k=+0}^{n} (2)^k$$

$$\boxed{\sum_{k=0}^{N} a^k = \frac{a^{n+1} - 1}{a-1}}$$

$$\therefore h(n) = (1/4)^n \cdot \frac{2^{n+1} - 1}{2-1}$$

$$= (1/4)^n [2^{n+1} - 1]$$

2. Determine the output of the LTI system whose input and unit sample response are given as follows. (3)

$$x(n) = b^n u(n) \quad h(n) = a^n u(n)$$

$$y(n) = x(n) * h(n)$$

$$= \sum_{k=-\infty}^{\infty} x(k) . h(n-k)$$

$$= \sum_{k=-\infty}^{\infty} b^n u(k) . a^{(n-k)} u(n-k)$$

$$u(k) = \begin{cases} 1 & ; k \geq 0 \\ 0 & ; k < 0 \end{cases}$$

$\therefore$ lower limit changes to $k=0$ & $u(k) = 1$.

$$y(n) = \sum_{k=0}^{\infty} b^k . a^n . a^{-k} u(n-k)$$

using step response defination.

$$u(n-k) = \begin{cases} 1 & ; n \geq k \\ 0 & ; n < k. \end{cases}$$

$\therefore$ upper limit changes to n & $u(n-k) = 1$.

$$y(n) = \sum_{k=0}^{n} b^k a^n a^{-k}$$

$$= a^n \sum_{k=0}^{n} (ba^{-1})^k$$

$$y(n) = a^n \left[\frac{(ba^{-1})^{n+1} - 1}{ba^{-1} - 1} \right]$$

$$= a^n \cdot \frac{(b/a)^{n+1} - 1}{b/a - 1} = a^n \frac{(b/a)^{n+1} - 1}{b - a/a}$$

$$= a^n \cdot a \left[\frac{b^{n+1} - a^{n+1} / a^{n+1}}{b - a} \right]$$

$$= a^{n+1} \cdot a^{-n-1} \left[\frac{b^{n+1} - a^{n+1}}{b - a} \right]$$

$$= \frac{b^{n+1} - a^{n+1}}{b - a}.$$

* Properties of Impulse response representation:-

1. Causality of LTI System:-

A System is Said to be Causal if the o/p depends only on the present and past input.

this Condition can be expressed in terms of unit Sample response $h(n)$ for LTI system.

$$y(n) = \sum_{K=-\infty}^{\infty} h(K) \cdot x(n-K)$$

At $n = n_0$, the o/p $y(n_0)$ will be.

$$y(n_0) = \sum_{K=-\infty}^{\infty} h(K)\, x(n_0 - K)$$

write it interms of two seperate terms.

$$y(n_0) = \sum_{K=+0}^{\infty} h(K) x(n_0 - K) + \sum_{K=-\infty}^{-1} h(K)\, x(n_0 - K)$$

Expanding the Summation

$$y(n_0) = [h(0)x(n_0) + h(1)x(n_0-1) + h(2)x(n_0-2) + \cdots] + [h(-1)x(n_0+1) + h(-2)x(n_0-2) \cdots]$$

here $x(n_0)$ = present i/p

$x(n_0+1), x(n_0+2)$ = future i/p

$x(n_0-1), x(n_0-2)$ = past i/p

we know that o/p of casual system at $n = n_0$ depends on i/p for $n \leq n_0$

$\therefore h(-1) = h(-2) = h(-3) = \cdots = 0$

$x(n_0+1); x(n_0+2), \cdots = 0$

$\therefore$ A system is said to be causal if $h(n) = 0$ for $n < 0$.

2. Stability of LTI System :-

A System is said to be stable if it produces bounded o/p for every bounded i/p

- Derive stability criteria for LTI system.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

Taking absolute values on both sides.

$$|y(n)| = \left|\sum_{k=-\infty}^{\infty} h(k) \cdot x(n-k)\right|$$

absolute value of total sum is always less than or equal to sum of absolute values of individual terms.

$\therefore |y(n)| \le \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$

If the i/p sequence $x(n)$ is bounded, then there exist a finite number M_x, such that

$$|x(n)| \le M_x < \infty.$$

$$\therefore |y(n)| \le M_x \sum_{k=-\infty}^{\infty} |h(k)|$$

$\therefore$ for $|y(n)|$ to be finite $\sum_{k=-\infty}^{\infty} |h(k)| < \infty.$

$\therefore$ if the sum of impulse response is finite then o/p $y(n)$ is also finite.

$\Rightarrow$ $\boxed{\text{A system is said to be stable if } \sum_{k=-\infty}^{\infty} |h(k)| < \infty.}$

3. Memoryless and With Memory System :-

- A System is said to be memoryless if o/p depends only on the present i/p.

- By the defination of Convolution.

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\,x(n-k)$$

on expanding.

$$y(n) = \dots + h(-3)\,x(n+3) + h(-2)\,x(n+2) + h(-1)\,x(n+1) + h(0)\,x(n) + h(1)\,x(n-1) + \dots$$

$\therefore$ from the above equation, if the system is memoryless then $y(n) = h(0)x(n)$ and rest all term have to be zero.

this can also be written as $h(n) = 0$ for $n \neq 0$

$\therefore$ Condition for unit sample response of memoryless system is

$$\boxed{h(n) = C.\delta(n)}$$

"C" = arbitrary constant.

Eg 1. Find the step response for impulse response.

$$h(n) = (1/2)^n u(n)$$

$$S(n) = \sum_{K=-\infty}^{n} h(K)$$

$$\Rightarrow S(n) = \sum_{K=-\infty}^{n} (1/2)^K u(K)$$

$$= \sum_{K=0}^{n} (1/2)^K . 1$$

$$\boxed{\sum_{K=0}^{n} a^K = \frac{a^{n+1} - 1}{a - 1}}$$

$$= \frac{(1/2)^{n+1} - 1}{(1/2 - 1)}$$

2. $h(n) = (1/2)^n u(n-3)$

$$S(n) = \sum_{K=-\infty}^{n} (1/2)^K u(K-3)$$

$$= \sum_{K=3}^{n} (1/2)^K$$

K should be = 0 to apply the standard equation.

$\therefore\ k-3=p \qquad k=3 \qquad k=n$

$k=p+3 \qquad p=0 \qquad p=n-3$

$$\Rightarrow s(n)=\sum_{p=0}^{n-3}(1/2)^{p+3}$$

$$=(1/2)^3\sum_{p=0}^{n-3}(1/2)^{p}$$

$$=(1/2)^3\left[\frac{(1/2)^{n-3+1}-1}{(1/2)-1}\right]$$

$$=(1/2)^3\left[\frac{(1/2)^{n-2}-1}{(1/2)-1}\right]$$

3. $h(t)=t\,u(t)$

$$s(t)=\int_{\tau=-\infty}^{t}h(\tau)\,d\tau$$

$$=\int_{-\infty}^{t}\tau\cdot u(\tau)\,d\tau$$

$$=\int_{0}^{t}\tau\,d\tau=\frac{t^2}{2}.$$

4. $h(t)=e^{-at}u(t)$

$$s(t)=\int_{\tau=-\infty}^{t}e^{-a\tau}u(\tau)\,d\tau$$

$$=\int_{0}^{t}e^{-a\tau}\,d\tau=\left[\frac{e^{-a\tau}}{-a}\right]_0^t=\frac{e^{-at}}{-a}-\frac{e^{0}}{-a}=-1/a\left[e^{-at}-1\right]$$

* Differential / Difference Equation representation for LTI System :- ⑥

- this is a type of time domain representation for LTI system.
- It gives the relationship between i/p & o/p of LTI system.
- Differential equation are used to represent Continuous time system.
- Difference equation are used to represent discrete time system.

1. Differential equation representation for Continuous time LTI System :-

general form of a linear Constant - coefficient differential equation is given by.

$$\sum_{K=0}^{N} a_K \cdot \frac{d^K}{dt^K} y(t) = \sum_{K=0}^{M} b_K \frac{d^K}{dt^K} x(t) \quad -① $$

where $x(t)$ is i/p of the system.

$y(t)$ is o/p of the system.

N, M = Order of differential equation.

Solution for differential equation

the expression for the o/p $y(t)$ has 2 component.

(zero i/p response)

1. Natural response: o/p obtained due to initial Condition. denoted by $y^{(n)}(t)$

2. forced response :- o/p obtained due to i/p condition denoted by $y^{(f)}(t)$.

[Zero state response)

$\therefore$ natural response is the system o/p with no input.

forced response is the system o/p for zero initial condition.

Complete response of a system is equal to sum of natural response and forced response.

1. Natural Response $y^{(n)}(t)$

for natural response only initial conditions have to be considered and i/p is zero.

$\therefore$ the equation reduces to.

$$\sum_{K=0}^{N} a_K \frac{d^K}{dt^K} y(t) = 0 \qquad — (2)$$

this is called a homogenous differential equation.

$\therefore$ Natural response of a system is of the form.

$$y^{(n)}(t) = \sum_{i=1}^{N} C_i \, e^{r_i . t} \qquad — (3)$$

r_i are the N roots of the system characteristic equation. which is given as.

$$\sum_{K=0}^{N} a_K . r^K = 0 \qquad — (4)$$

Substitute the roots in eq (3) we get natural response.

① Find the natural response for the System described by the differential equation. (7)

$$5.\frac{dy(t)}{dt} + 10y(t) = 2x(t) \qquad y(0) = 3.$$

find the natural response, we have to consider $x(t) = 0$.

$$\therefore 5\frac{dy(t)}{dt} + 10y(t) = 0$$

To obtain characteristic equation replace $\frac{d^k y(t)}{dt^k} = r^k$.

$$\Rightarrow 5r + 10 = 0$$

$$r = -10/5$$

$$r = -2.$$

$\therefore$ natural response $y^{(n)}(t) = Ce^{-2rt}$

$$\therefore y^{(n)}(t) = Ce^{-2t}.$$

given $y(0) = 3$.

$$y(0) = Ce^{-2\times 0} = 3.$$

$$\therefore C = 3.$$

$\therefore$ natural response $y^{n}(t) = 3e^{-2t}.$

② Determine the natural response of the system described by differential equation -

$$10\frac{dy(t)}{dt} + 2y(t) = x(t) \quad \text{with } y(0) = 2.$$

natural response is obtained with zero i/p

$$\therefore 10\frac{dy(t)}{dt} + 2y(t) = 0.$$

To Obtain roots replace $\frac{d^k y(t)}{dt^k}$ by r^k.

$\therefore\ 10r_1 + 2 = 0$

$\Rightarrow r_1 = -2/10 = -0.2$

Roots are real.

To obtain natural response.

$$y^{(n)}(t) = \sum_{k=0}^{N} C_i e^{r_i t}$$

N = order of the characteristic equal.

$$= \sum_{K=0}^{1} C_i e^{r_1 t}$$

$$y^{n}(t) = C_1 e^{-0.2t}$$

To obtain value of constant in $y^{(n)}(t)$

initial condition is $y(0) = 2$ and $t = 0$.

$$\Rightarrow y^{n}(0) = C_1 e^{-0.2\times 0}$$

$$y^{n}(0) = 2 = C_1$$

$$\therefore \boxed{y^{n}(t) = 2e^{-0.2t}}$$

(3) find the natural response of the system described by the differential equation

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = x(t) + \frac{dx(t)}{dt}$$

with initial condition $y(0) = 0$ $\left.\frac{dy(t)}{dt}\right|_{t=0} = 1$.

Natural response is obtained with zero i/p.

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = 0.$$

To obtain characteristic equation replace $y(t)$ by r^k.

$$y''(t) + 3y'(t) + 2y(t) = 0$$

$$r^2 + 3r + 2 = 0.$$

find the roots of characteristic equation.

$$r^2 + 3r + 2 = 0$$

$$r^2 + 1r_1 + 2r_2 + 2 = 0$$

$$r(r_1+1) + 2(r_1+1) = 0$$

$$(r_1+1)(r_2+2) = 0$$

$r+1 = 0$ $\quad$ $r+2 = 0$

$r_1 = -1$ $\quad$ $r_2 = -2.$

Both the roots are real.

To obtain natural response.

$$y^{(n)}(t) = \sum_{K=1}^{N} C_i e^{r_i t} \qquad N = 2.$$

$$y^n(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t} \quad - ①$$

$$\frac{dy^n(t)}{dt} = C_1 e^{-t} + C_2 e^{-2t}$$

$$= -C_1 e^{-t} - 2C_2 e^{-2t} \quad - ②.$$

apply initial condition.

$y(0) = 0$, $\frac{dy^n(t)}{dt^n} = 1.$

$$C_1 + C_2 = 0$$
$$-C_1 - 2C_2 = 1.$$

Solving Simultaneous equation

$$C_1 = 1 \qquad C_2 = -1$$

$\therefore$ natural response $\boxed{y^n(t) = e^{-t} - e^{-2t}}$

4. Determine the natural response for the system described by the following differential equation.

$$\frac{d^2y(t)}{dt^2} + 4y(t) = 3\frac{dx(t)}{dt} \quad \text{with } y(0) = -1,$$

$$\left.\frac{dy(t)}{dt}\right|_{t=0} = 1. \text{ or } y'(0) = 1.$$

Natural response of the System when i/p = 0.

$$\frac{d^2y(t)}{dt^2} + 4y(t) = 0$$

Characteristic equation is obtained by replacing r

$$\therefore r^2 + 4 = 0.$$
$$r^2 = -4$$
$$\therefore r_1 = \pm 2j$$

roots are non-repeated and purely imaginary, natural response is of the form.

$$y^n(t) = C_1 \cos 2t + C_2 \sin 2t$$

$$\therefore \frac{dy^n(t)}{dt} = -2C_1 \sin 2t + 2C_2 \cos 2t$$

Put $y(0) = -1$.

$$y(0) = C_1 \cos 0 + C_2 \sin 0.$$

$$\boxed{-1 = C_1}$$

Put $\left.\frac{dy(t)}{dt}\right|_{t=0} = \dot{y}(0) = 1$.

$$1 = 2C_2. \quad \boxed{C_2 = 1/2}$$

∴ natural response

$$y^{(n)}(t) = -\cos 2t + 1/2 \sin 2t.$$

* Forced Response :- $y^{(f)}(t)$

It is the Solution of differential equation for the given i/p with initial Condition equal to Zero.

It has two Components.

1. a term resembling the natural response $y^{(n)}(t)$ &
2. a particular Solution $y^{(P)}(t)$.

$y^{(P)}(t)$ is obtained by assuming the System o/p to have the Same form as i/p.

Eg $x(t) = Ae^{-at}$ then we assume $y^{(P)}(t) = Ke^{-at}$

constant K is determined in Such a way that $y^{(P)}(t)$ Satisfies the System differential equation.

1. Determine the forced response for the given System

$$5\frac{dy(t)}{dt} + 10y(t) = 2x(t) \quad \text{with i/p } x(t) = 2u(t)$$

forced response is due to i/p only. It has 2 terms.

1. term resemble natural response
2. Particular Solution

$\therefore y^{(f)}(t) = y^{(n)}(t) + y^{(P)}(t)$

1. find $y^{(n)}(t)$:-

$\Rightarrow 5 \cdot \frac{dy(t)}{dt} + 10y(t) = 0.$

characteristic equation is obtained by replacing r^{k}.

$\Rightarrow 5r + 10 = 0.$

$r = -2.$

$\therefore \boxed{y^{n}(t) = C_1 e^{-2t}}$

2. Particular Solution is of the form of i/p $x(t)$

$x(t) = 2u(t) \rightarrow$ constant

$\therefore y^{(P)}(t) = K.$

$\therefore 5\frac{dy(t)}{dt} + 10y(t) = 2x(t)$

$5\frac{dK}{dt} + 10K = 2[2u(t)]$

$5 \cdot 0 + 10K = 4u(t)$

$10K = 4(1)$

$\boxed{K = 2/5}$

$\therefore$ forced response $y^{(f)}(t) = y^{(n)}(t) + y^{(P)}(t)$

$\boxed{y^{f}(t) = C_1 e^{-2t} + 2/5.}$

3. To find forced response :-

We assume initial condition are zero. $y(0) = 0$

$\therefore y(0) = C_1 e^{-2t} + 2/5$

$0 = C_1 + 2/5$

$C_1 = -2/5$

$\therefore y^{(f)}(t) = -2/5\, e^{-2t} + 2/5$

$\boxed{y^{(f)}(t) = 2/5\,(1 - e^{-2t})}$

2. Find the forced response for the system given by. (10)

$$5 \frac{dy(t)}{dt} + 10y(t) = 2x(t) \quad \text{with i/p } x(t) = e^{-t} u(t).$$

$$y^{(f)}(t) = y^{(n)}(t) + y^{(p)}(t)$$

1. To find $y^{(n)}(t)$ – i/p $x(t) = 0$.

$$\therefore 5\dot{y}(t) + 10y(t) = 0$$

replacing $\dot{y}(t)$ by r^k we have.

$$\Rightarrow 5r + 10 = 0.$$

$$r = -2.$$

$$\therefore y^{(n)}(t) = C_1 e^{-2t}$$

2. To find $y^{(p)}(t) \rightarrow$ o/p in the form of i/p.

$$x(t) = e^{-t} u(t)$$

$$y^{(p)}(t) = K e^{-t}$$

$$\therefore 5 \frac{dy^{(p)}(t)}{dt} + 10y^{(p)}(t) = 2x(t)$$

$$\Rightarrow -5Ke^{-t} + 10Ke^{-t} = 2 \cdot e^{-t} u(t)$$

$$-5Ke^{-t} + 10Ke^{-t} = 2e^{-t}$$

$$-5K + 10K = 2.$$

$$K = 2/5$$

3. find forced response:-

$$\therefore y^{(f)}(t) = y^{(n)}(t) + y^{(p)}(t)$$

$$\Rightarrow y^{(f)}(t) = C_1 e^{-2t} + 2/5 \, e^{-t}$$

Assume initial condition $y(0) = 0$.

$$0 = C_1 + 2/5$$

$$C_1 = -2/5$$

* Difference equation representation for discrete time system (11)

general form is given by.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

where $x(n)$ is the i/p to the system.

$y(n)$ is the o/p of the system.

N and M are the order of difference equation.

Solution of difference equation has 2 components.

1. Natural response – o/p associated with initial condition. denoted by $y^{(n)}(n)$.
2. the forced response – o/p due to only i/p denoted by $y^{(f)}(n)$

* Natural response :- $y^{(n)}(n)$.

o/p of a system with zero i/p

∴ general form reduces to.

$$\sum_{k=0}^{N} a_k y(n-k) = 0.$$ [homogenous difference equation]

∴ Natural response for discrete time system is of the form.

$$y^{(n)}(n) = \sum_{i=1}^{N} c_i r_i^n.$$

where r_i are the N roots of characteristic equation

Note: 1. If any roots r_i repeats p times, then we include p distint terms in natural response

i.e $r_i^n, n r_i^n, n^2 r_i^n, n^3 r_i^n \dots n^{p-1} r_j^n$.

2. the nature of each term in the natural response depends on the roots of the characteristic equation.

If 'r_i' are real, - natural response Consist of exponential terms.

r_i = imaginary - Sinusoidal terms.

r_i = Complex - exponentially damped Sinusoidal.

Eg 1. find the natural response for the system described by the following difference equation:

$$y(n) - \frac{9}{16} y(n-2) = x(n-1) \text{ with } y(-1) = 1$$

$$y(-2) = -1.$$

Sol: the given System is of order 2.

$\therefore$ 2 initial Condition.

To find natural response $x(n) = 0$.

$$\therefore y(n) - \frac{9}{16} y(n-2) = 0. \quad -(1)$$

To obtain characteristic equation replace $y(n-k) = r^k$

$$1 - \frac{9}{16} r^{-2} = 0. \Rightarrow 1 = \frac{9}{16} \cdot \frac{1}{r^2}$$

$$r^2 = 9/16 \Rightarrow r^2 = \pm 3/4$$

$r_1 = 3/4, \quad r_2 = -3/4$

$\therefore$ natural response is of the form.

$$y^{(n)}(n) = C_1(r_1)^n + C_2(r_2)^n$$
$$= C_1(3/4)^n + C_2(-3/4)^n$$

from the equation (1)

$$y(n) = \frac{9}{16}y(n-2) = C_1(3/4)^n + C_2(-3/4)^n = y^n(n)$$

Apply initial condition.

$$y(0) - \frac{9}{16}y(-2) = \frac{9}{16}(-1) = C_1 + C_2$$

$$\Rightarrow C_1 + C_2 = -9/16 \quad -(2)$$

$$y(1) = \frac{9}{16}y(-1) = 3/4\, C_1 - 3/4\, C_2 \quad -(2)$$

$$\Rightarrow 3/4\, C_1 - 3/4\, C_2 = 9/16 \quad -(3)$$

$$C_1 + C_2 = -9/16$$
$$3/4 C_1 - 3/4 C_2 = 9/16$$

$$3/4\, C_1 + \cancel{3/4}\, C_2 = -27/64$$
$$3/4\, C_1 - 3/4\, C_2 = 9/16$$

$$6/4\, C_1 = \frac{-27}{64} + \frac{9}{16} = \frac{-432 + 576}{1024}$$

$$6/4\, C_1 = \frac{144}{1024}$$

$$C_1 = \frac{144}{1024} \times \frac{4}{6} = \frac{24}{256} = 3/32.$$

$$3/32 + C_2 = -\frac{9}{16}$$

$$C_2 = \frac{-9}{16} - \frac{3}{32}$$

$$C_2 = \frac{-288 - 48}{512}$$

$$C_2 = \frac{-336}{512}$$

$$C_2 = \frac{-21}{32}.$$

$\therefore$ natural response

$$y^{(n)}(n) = \frac{3}{32}(3/4)^n - \frac{21}{32}(3/4)^n$$

* forced response $y^{(f)}(n)$:-

It is a solution to difference equation for the given i/p with initial condition as zero.

It has two component.

1. a term resembling the natural response $y^n(n)$
2. particular solution $y^p(n)$

A particular solution is obtained by assuming the system o/p has the same form as i/p.

$x(n)$	$y^p(n)$
A	K
$A\alpha^n$	$K\alpha^n$
$A\cos(\Omega n+\phi)$	$K_1\cos(\Omega n) + K_2\sin(\Omega n)$

Eg find the forced response for the system given by difference equation

$$y(n) - 1/4\, y(n-1) - \frac{1}{8} y(n-2) = x(n) + x(n-1)$$

with i/p $x(n) = (1/8)^n u(n)$

Sol: given: $y(n) = 1/4\, y(n-1) + 1/8\, y(n-2) + x(n) + x(n-1)$

$$y^f(n) = y^n(n) + y^p(n)$$

To obtain $y^n(n)$.

To obtain characteristic equation $y(n-k) = r$

$$1 - \tfrac{1}{4} r^{-1} - \tfrac{1}{8} r^{-2} = 0$$

$$r^2 - \frac{1}{4} r^1 - \frac{1}{8} = 0.$$

$$r_1 = -\tfrac{1}{4} \qquad r_2 = \tfrac{1}{2}.$$

$$\therefore\ y^h(n) = C_1 (-\tfrac{1}{4})^n + C_2 (\tfrac{1}{2})^n$$

To obtain particular solution :-

$$x(n) = (\tfrac{1}{8})^n u(n).$$

$$\therefore\ y^p(n) = K (\tfrac{1}{8})^n u(n)$$

Substituting it in given equation we have.

$$K(\tfrac{1}{8})^n u(n) - \tfrac{1}{4} K (\tfrac{1}{8})^{n-1} u(n-1) - \tfrac{1}{8} K (\tfrac{1}{8})^{n-2} u(n-2) =$$

$$(\tfrac{1}{8})^n u(n) + (\tfrac{1}{8})^{n-1} u(n-1)$$

$$\Rightarrow K(\tfrac{1}{8})^n - \tfrac{1}{4} K (\tfrac{1}{8})^{n-1} - \tfrac{1}{8} K (\tfrac{1}{8})^{n-2} = (\tfrac{1}{8})^n + (\tfrac{1}{8})^{n-1}$$

divide through out by $(\tfrac{1}{8})^{-n}$.

$$\Rightarrow K - \tfrac{1}{4} K (\tfrac{1}{8})^{-1} - \tfrac{1}{8} K (\tfrac{1}{8})^{-2} = 1 + (\tfrac{1}{8})^{-1}$$

$$K[1 - 2 - 8] = 9$$

$$K = -1$$

$$\therefore\ y^f(n) = C_1 (-\tfrac{1}{4})^n + C_2 (\tfrac{1}{2})^n - (\tfrac{1}{8})^n$$

Assuming initial condition $y(-1) = 0$ $y(-2) = 0$.

* Complete Response : $y(n)$

- total response of the System is the Sum of the natural response and forced response.
- To determine the complete response we must take both the initial Condition and i/p into Consideration.

Q1. Find the response of the System described by the difference equation $y(n) - \frac{1}{9} y(n-2) = x(n-1)$ with $y(-1) = 1$ & $y(-2) = 0$ and $x(n) = u(n)$.

given :- $y(n) - \frac{1}{9} y(n-2) = x(n-1)$

$$y(n) = \frac{1}{9} y(n-2) + x(n-1)$$

$$y(n) = y^f(n) + y^n(n)$$

To obtain characteristic equation replace $y(n-k)$ by r^{-k}.

$$1 - \frac{1}{9} r^{-2} = 0$$

$$r^2 - \frac{1}{9} = 0.$$

$$r^2 = \frac{1}{9}. \qquad \therefore r_1 = \frac{1}{3}, \quad r_2 = -\frac{1}{3}$$

$$\therefore y^n(n) = C_1 (\tfrac{1}{3})^n + C_2 (-\tfrac{1}{3})^n.$$

Since $x(n) = u(n)$ $\therefore$ partial Solution is of the form.

$$y^P(n) = K.u(n)$$

Substitute in the given equation

$$K u(n) - \frac{1}{9} K u(n-2) = u(n-1)$$

$$K - \frac{1}{9} K = 1.$$

$K[1-1/9] = 1.$

$K[8/9] = 1.$

$\boxed{K = 9/8.}$

$\therefore y(n) = C_1(1/3)^n + C_2(-1/3)^n + 9/8$

Apply initial condition $y(-1) = 1$ and $y(-2) = 0.$

$\therefore y(n) = \frac{1}{9} y(n-2) + x(n) = C_1(1/3)^n + C_2(-1/3)^n + 9/8$

$y(0) = 1/9\, y(-2) + \cancel{x}(-1) = C_1(1/3)^0 + C_2(-1/3)^0 + 9/8.$

$\Rightarrow C_1 + C_2 + 9/8 = 0 \quad -(1)$

$y(1) = 1/9\, y(-1) + x(0) = C_1(1/3)^1 + C_2(-1/3)^1 + 9/8.$

$1/9 + 1 = 1/3\, C_1 - 1/3\, C_2 + 9/8.$

$10/9 = 1/3\, C_1 - 1/3\, C_2 + 9/8 \quad -(2)$

$1/3\, C_1 - 1/3\, C_2 + 9/8 = 10/9.$

$C_1 + C_2 + 9/8 = 0 \quad \times 1/3.$

$1/3\, C_1 - \cancel{1/3\, C_2} + 9/8 = 10/9$

$1/3\, C_1 + \cancel{1/3\, C_2} + 9/24 = 0$

$C_1(2/3) + [9/8 + 9/24] = \frac{10}{9}.$

$C_1(2/3) + \left[\frac{27+9}{24}\right] = \frac{10}{9}$

$C_1(2/3) + \frac{36}{24} = \frac{10}{9}$

$C_1(2/3) = 10/9 - 3/2$

$$c_1(2/3) = \frac{20-27}{18} = -7/18$$

$$c_1 = -7/18 \times \frac{3}{2}$$

$$\boxed{c_1 = -\frac{7}{12}}$$

$$\boxed{c_2 = -\frac{13}{24}}$$

$$c_1 + c_2 + 9/8 = 0$$

$$c_2 = -9/8 + \frac{7}{12}$$

$$c_2 = \frac{-108+56}{96}$$

$$c_2 = -\frac{52}{96} = -\frac{13}{24}$$

$$\therefore y(n) = -7/12\,(1/3)^n - \frac{13}{24}\,(1/3)^n + 9/8.$$

* Block diagram representation :-

- It is a pictorial representation which describes different set of internal computation used to determine the o/p from i/p.

- It has a great significance as it helps in the implementation of the system using computer.

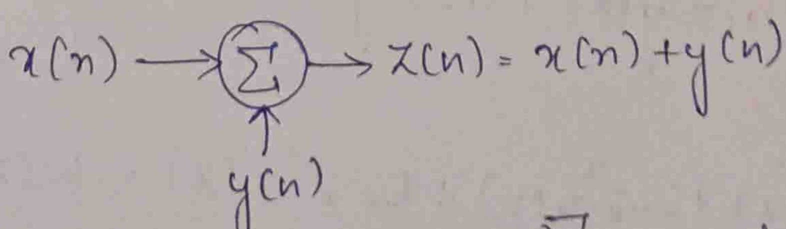
1. for discrete - time system :-

~~s~~ elementary operation.

a. Scalar multiplication

$$x(n) \xrightarrow{a} y(n) = ax(n)$$

b. Addition



Σ can also be written as +

3. time shift (Delay elements)

$$x(n) \rightarrow \boxed{z^{-1}} \rightarrow y(n) = x(n-1)$$

$$x(n) \rightarrow \boxed{z^{-k}} \rightarrow y(n) = x(n-k)$$

4. time advance element

$$x(n) \rightarrow \boxed{z} \rightarrow y(n) = x(n+1)$$

$$x(n) \rightarrow \boxed{z^{k}} \rightarrow y(n) = x(n+k)$$

- A discrete time System can be realized in different ways here we will have 4 types of relization

Direct form {
1. Direct form 1.
2. Direct form 2.
}
3. Cascade form.
4. Parallel form

- Direct form II will be generally used as it is the advance form Direct for 1.

* Direct form 1:-

Consider a System with System function.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \quad \begin{matrix} - \text{Zero's} \\ = \text{Poles.} \end{matrix}$$

(Transfer function)

$$\therefore Y(z) + a_1 z^{-1} Y(z) + a_2 z^{-2} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z)$$

Let $b_0 X(z) + b_1 z^{-1} X(z) + b_2 z^{-2} X(z) = W(z)$

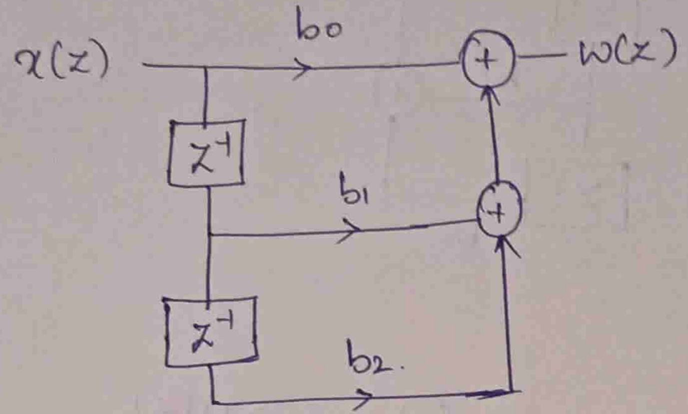
Step 1 → Realize the Zero's.

$\therefore$ $y(z) + a_1 z^{-1} y(z) + a_2 z^{-2} y(z) = w(z)$

$w(z) = b_0 x(z) + b_1 z^{-1} x(z) + b_2 z^{-2} x(z)$

On representing $w(z)$:-

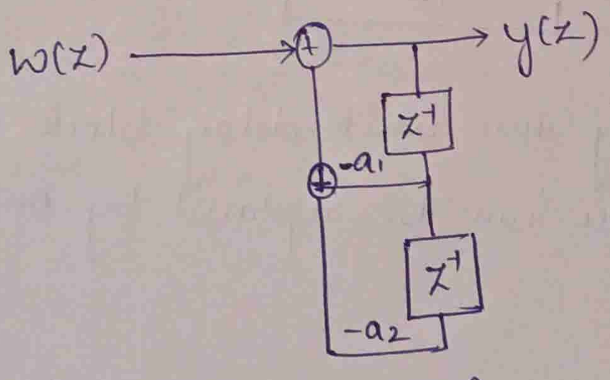
when ever we realize zero's we get forward path.



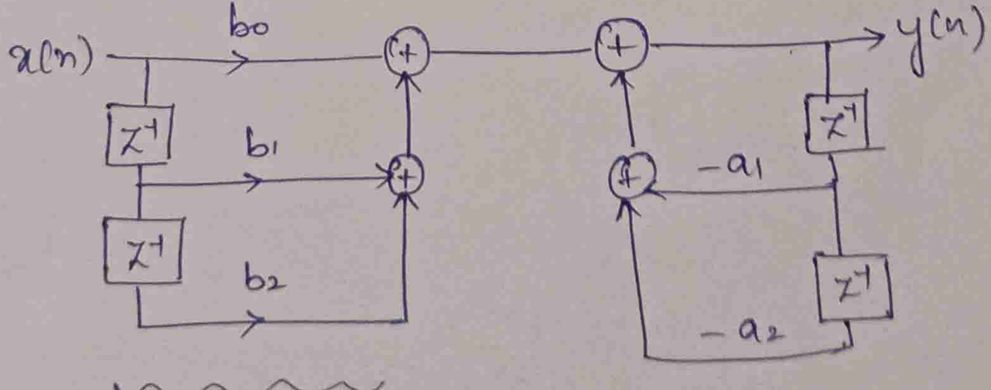
Step2: realization of poles.

$y(z) = w(z) - \underbrace{a_1 z^{-1} y(z) - a_2 z^{-2} y(z)}$

portion of o/p $\therefore$ we get feedback forms.



Combine the realization of zero's and poles.



Zero's

forward, coefficient sign does not change

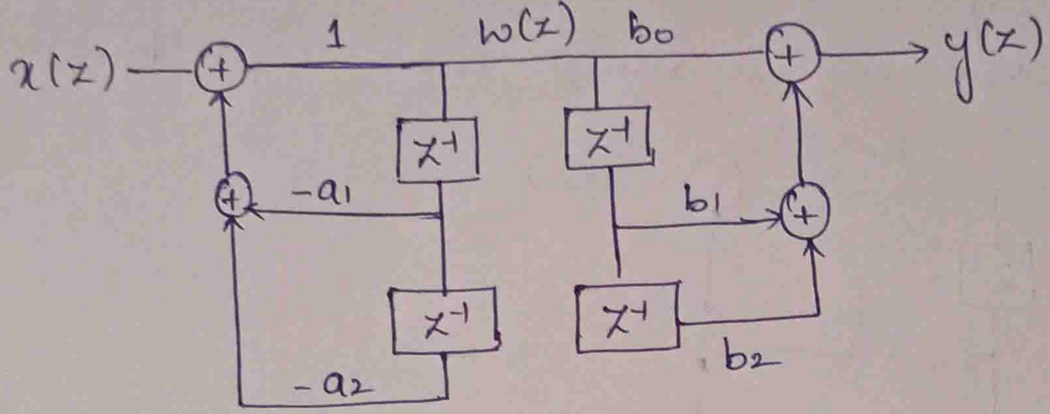
Poles → feed back

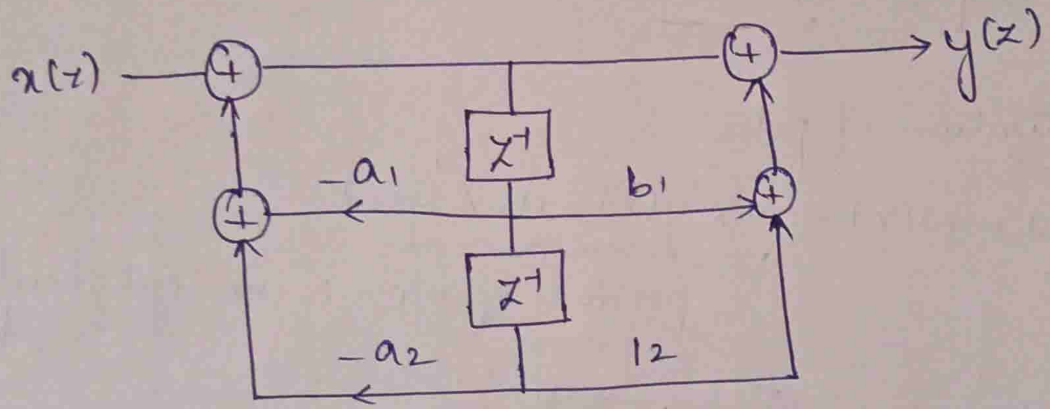
Co-eff sign changes.

* Direct form - II

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \begin{matrix} = \text{zero.} \\ = \text{poles} \end{matrix}$$

Step 1 : Realize the poles.





w(n) is delayed by two unit delay block to produce w(n-1) hence can be replaced by one block.

Let us consider the difference equation

$$\sum_{K=0}^{N} a_K y(n-K) = \sum_{K=0}^{M} b_K x(n-K)$$

Assume the order of the system to be $N = M = 2$.

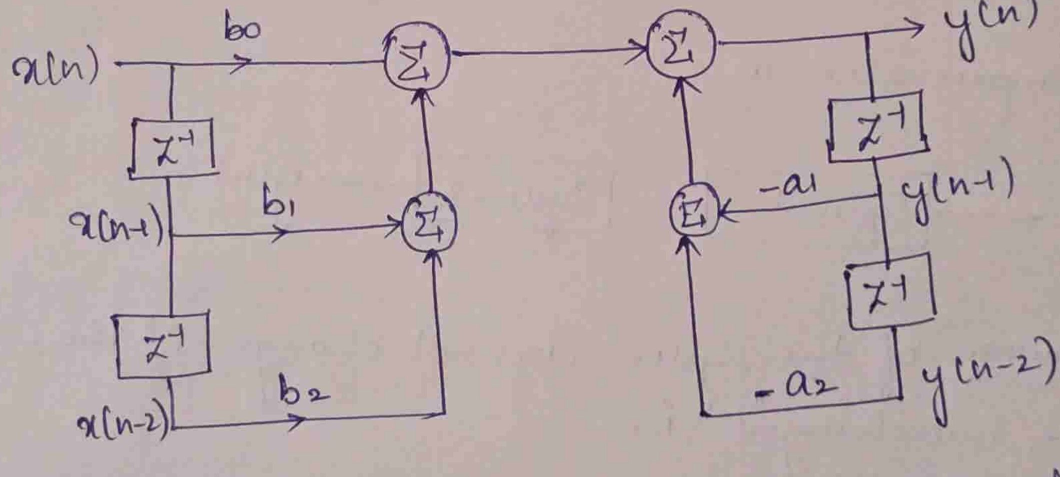
$$\sum_{K=0}^{2} a_K y(n-K) = \sum_{K=0}^{M=2} b_K x(n-K)$$

on expanding we have

$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

Put $a_0 = 1$ in the above equation then.

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

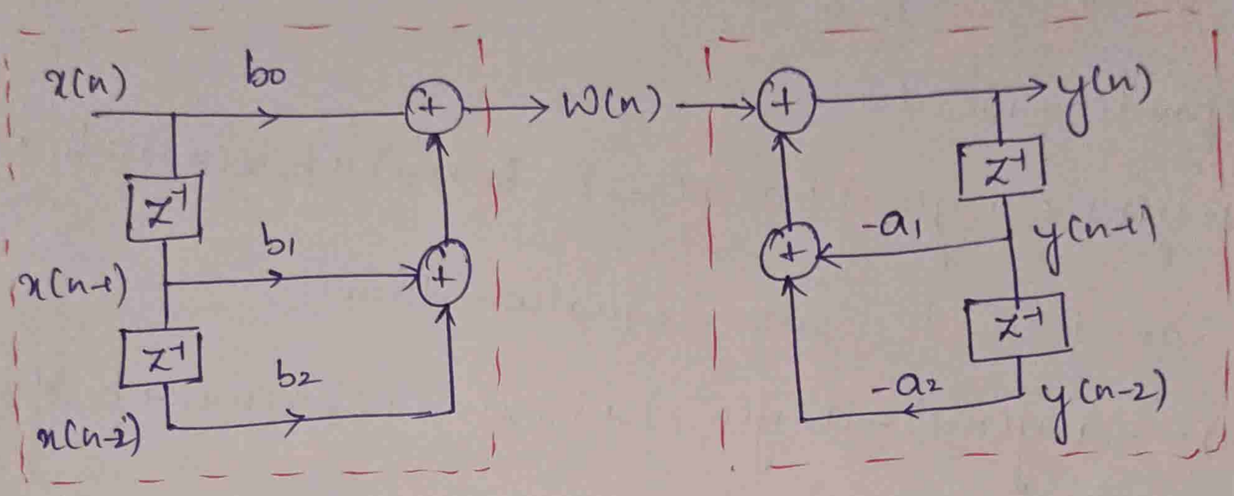


- i/p is given to the unit delay block. Its o/p is $x(n-1)$ which is given as i/p to the next delay block.
- b_0, b_1, b_2 are multiplier.
- Similarly o/p delay block is represented.

Direct form - II implementation :-

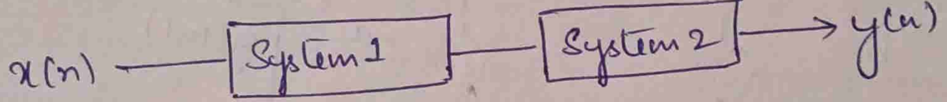
In direct form 1 two systems are cascaded.

1. Second order System corrosponds to present and past i/p system.
2. System corrosponds to past o/p system.

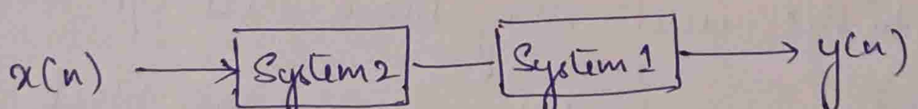


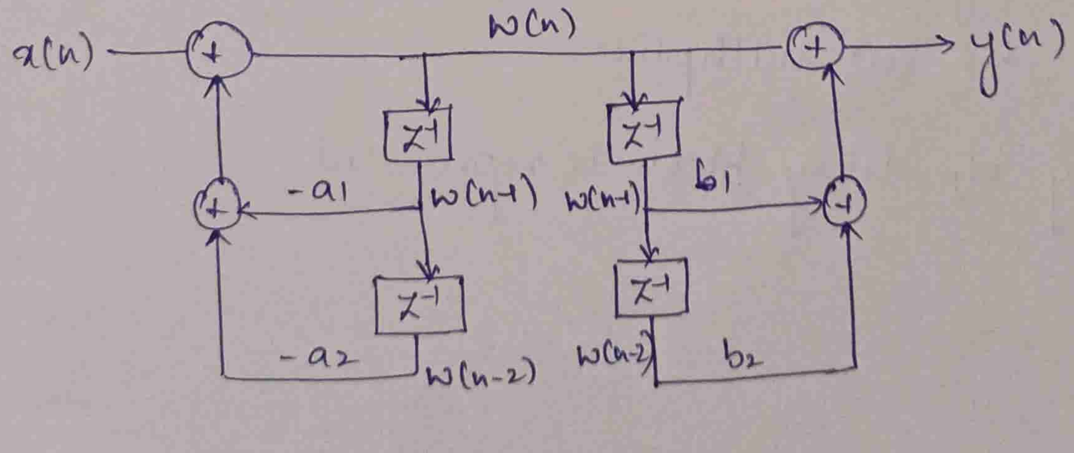
System 1 System 2.

It can be represented as.



the performance of the System does not change if the Systems are interchanged i.e.



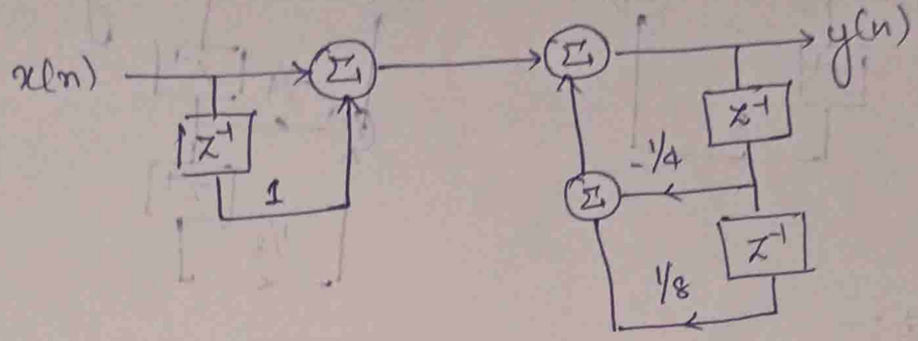


(1a)

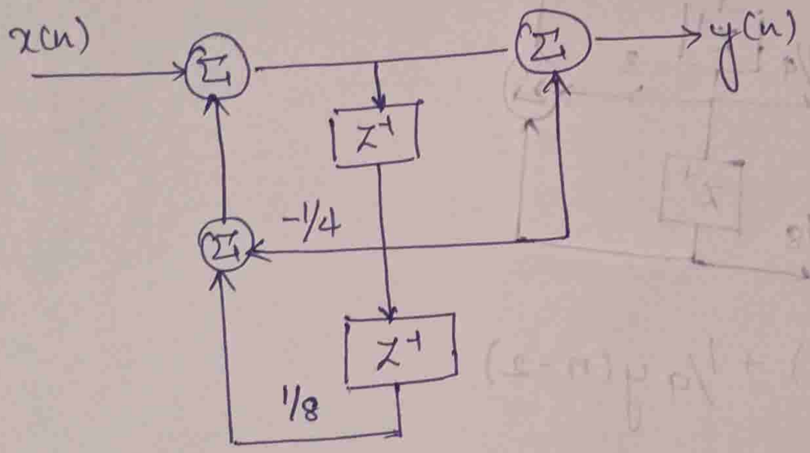
① Draw the direct form I and direct form II implementation for the system described by

$$y(n) + \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = x(n) + x(n-1)$$

Sol: Direct form I

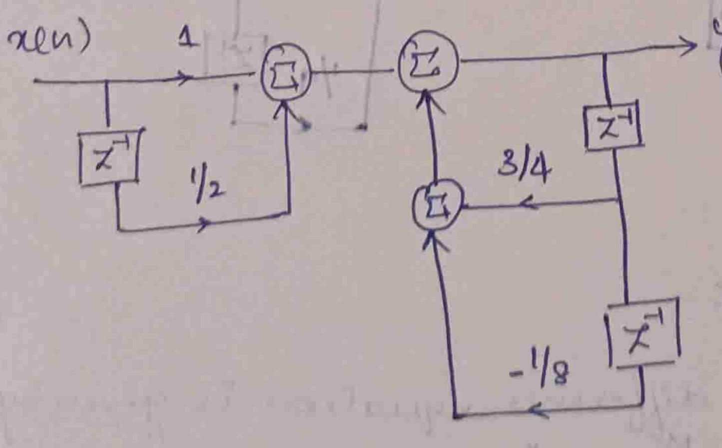


Direct form II

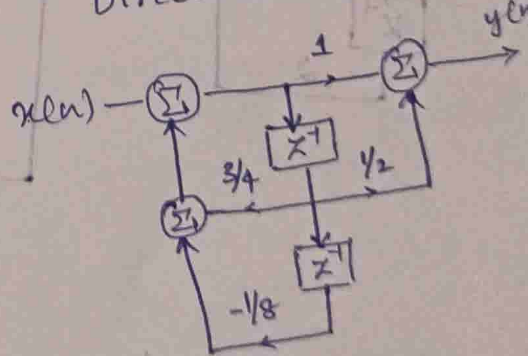


② $y(n) = \frac{3}{4} y(n-1) - \frac{1}{8} y(n-2) + x(n) + \frac{1}{2} x(n-1)$

Direct form 1.



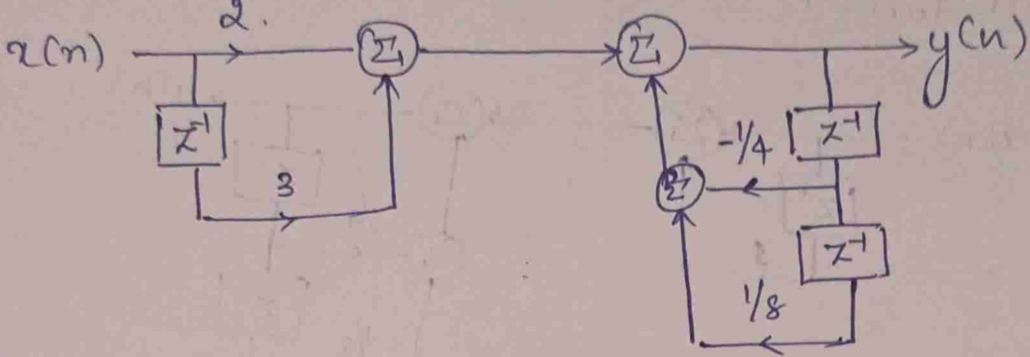
Direct form II



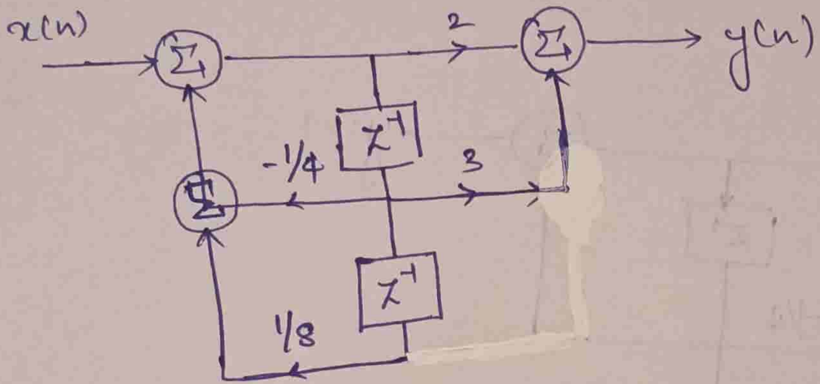
3. $y(n) + \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = 2x + 3x(n-1)$

$$y(n) = -\frac{1}{4} y(n-1) + \frac{1}{8} y(n-2) + 2x + 3x(n-1)$$

Direct form 1:-

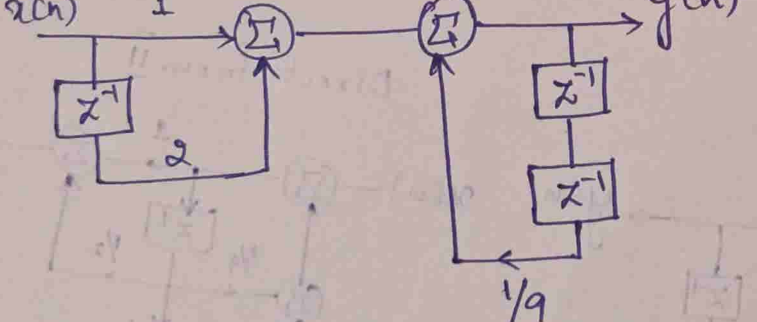


Direct form 2:-

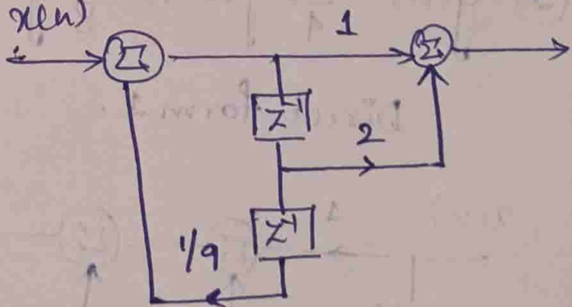


④ $y(n) = x(n) + 2x(n-1) + \frac{1}{9} y(n-2)$

Direct form 1:-



Direct form II



For Continuous - time system:-

the general form of difference equation is given by.

$$\sum_{K=0}^{N} a_K \frac{d^K y(t)}{dt^K} = \sum_{K=0}^{M} b_K \frac{d^K x(t)}{dt}$$

By integrating the above equation N times we get

$$\sum_{k=0}^{N} a_k\, y^{(N-k)}(t) = \sum_{k=0}^{M} b_k\, x^{(N-k)}(t)$$

Let's consider a second order system for which $a_2 = 1$.

$$a_0 y^2(t) + a_1 y'(t) + a_2 y(t) = b_2 x(t) + b_1 x'(t) + b_0 x^2(t)$$

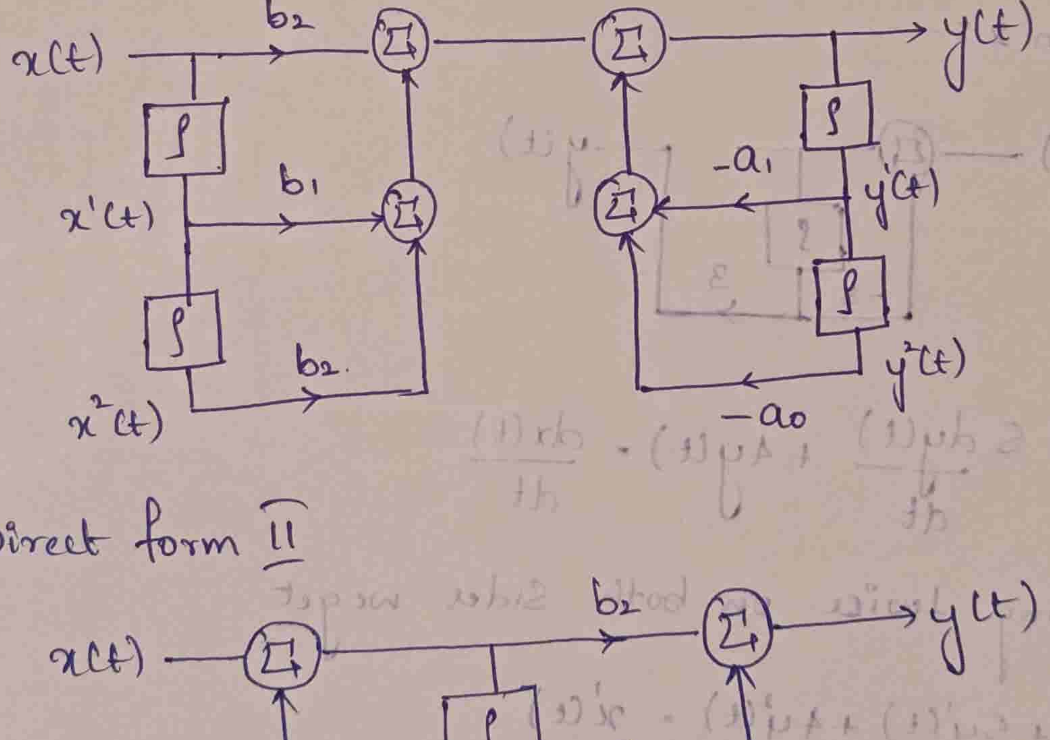
$$\therefore y(t) = b_2 x(t) + b_1 x'(t) + b_0 x^2(t) - a_0 y^2(t) + a_1 y'(t)$$

In general. $y'(t) \rightarrow$ 1 fold integral of $y(t)$

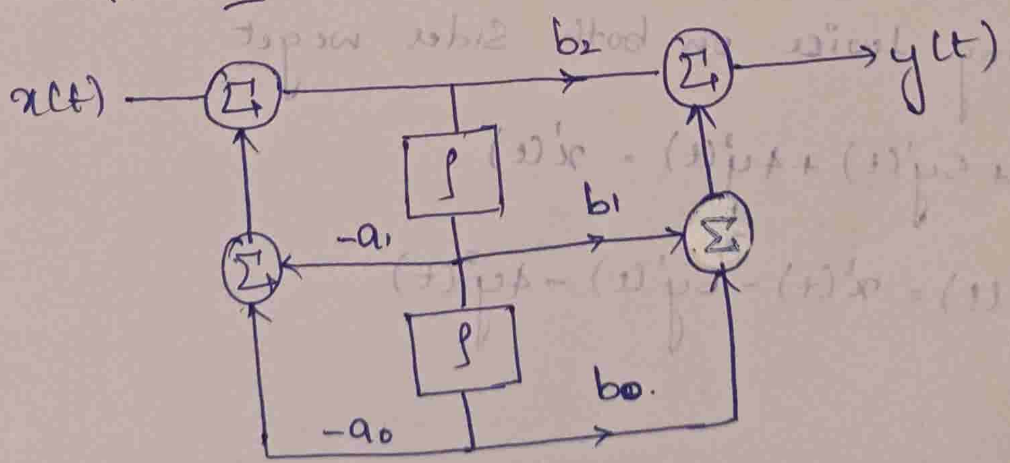
$y^2(t) \rightarrow$ 2 fold integral of $y(t)$

$y^m(t) \rightarrow$ m fold integral of $y(t)$

$\therefore$ Block diagram representation is as given below.



Direct form II



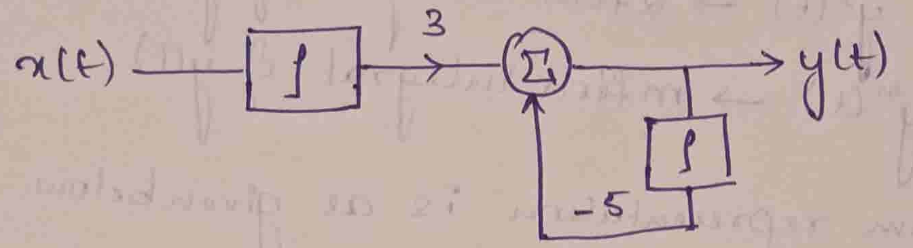
§ 1. By converting the differential equation to integral equation, draw direct form I and direct for II implementation for the system.

$$\frac{dy(t)}{dt} + 5y(t) = 3x(t)$$

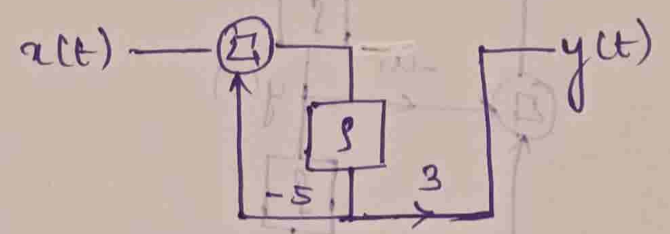
Integrating both sides we get

$$y(t) + 5y'(t) = 3x'(t)$$

Direct form 1:



Direct form 2:-

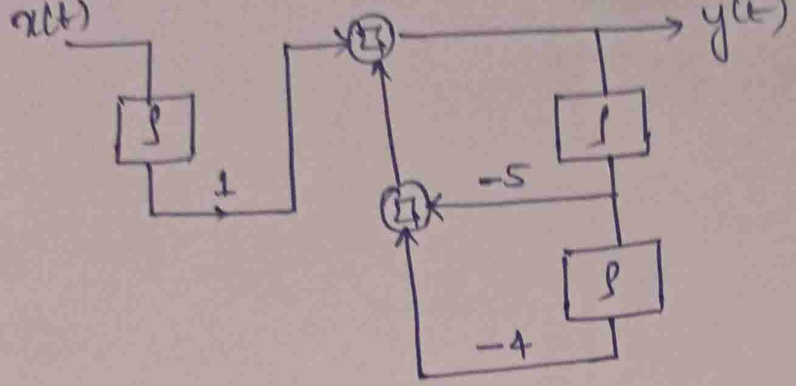


② $$\frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 4y(t) = \frac{dx(t)}{dt}$$

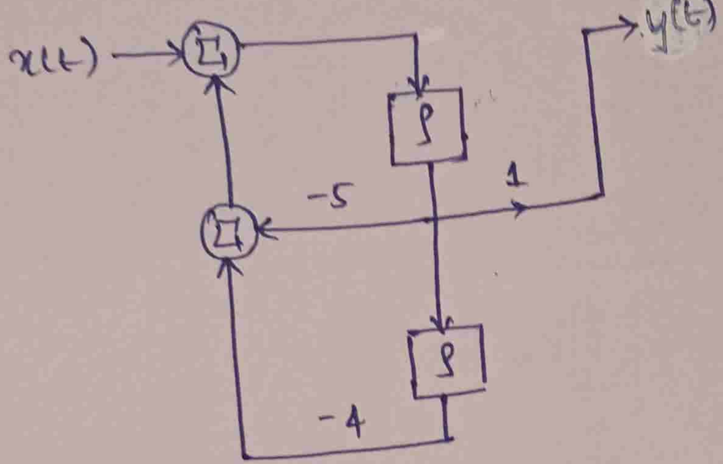
Integrating twice on both sides we get

$$y(t) + 5y'(t) + 4y^2(t) = x'(t)$$

$$\therefore\ y(t) = x'(t) - 5y'(t) - 4y^2(t)$$



Direct form II :-



* Cascade form :-

for the given function we need to represent it in terms of product terms.

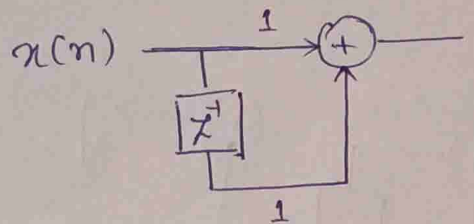
Eg $H(z) = H_1(z) * H_2(z)$

and each product term is represented in terms of direct form II / direct form I

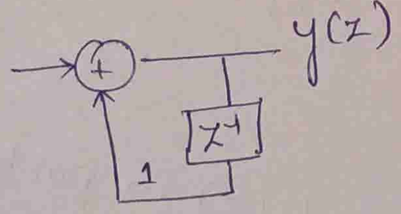
Eg $H(z) = \frac{1+z^{-1}}{1-z^{-1}} \Rightarrow (1+z^{-1}) \times \frac{1}{(1-z^{-1})} = \frac{Y(z)}{X(z)}$

$= [H_1(z)] \times [H_2(z)]$

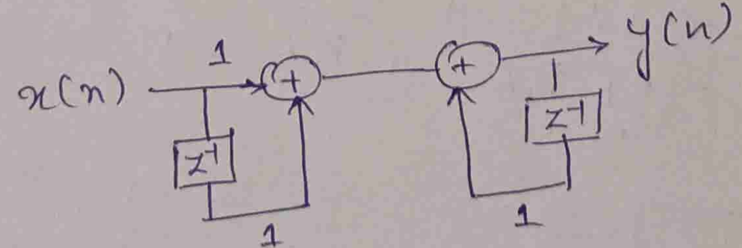
first we will realize $H_1(z) = 1+z^{-1}$

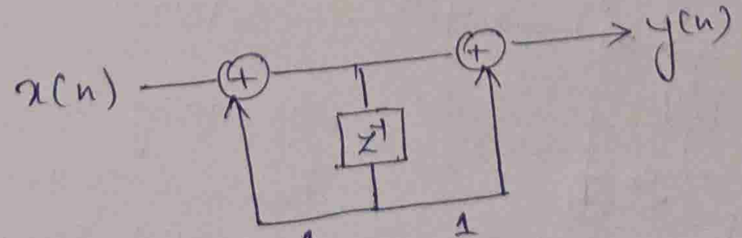


Realize $H_2(z) = \frac{1}{1-z^{-1}}$



Cascade is Serial Connection





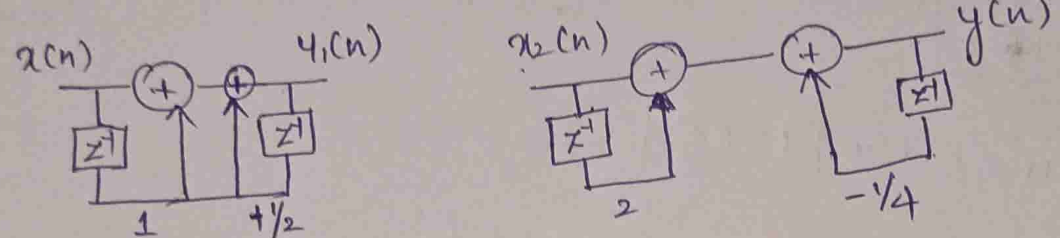
Eg. $H(z) = \dfrac{(1+z^{-1})(1+2z^{-1})}{(1-\frac{1}{2}z^{-1})(1+\frac{1}{4}z^{-1})} = \dfrac{Y(z)}{X(z)}$

We will assume. $\dfrac{(1+z^{-1})}{(1-\frac{1}{2}z^{-1})} \times \dfrac{(1+2z^{-1})}{(1+\frac{1}{4}z^{-1})}$

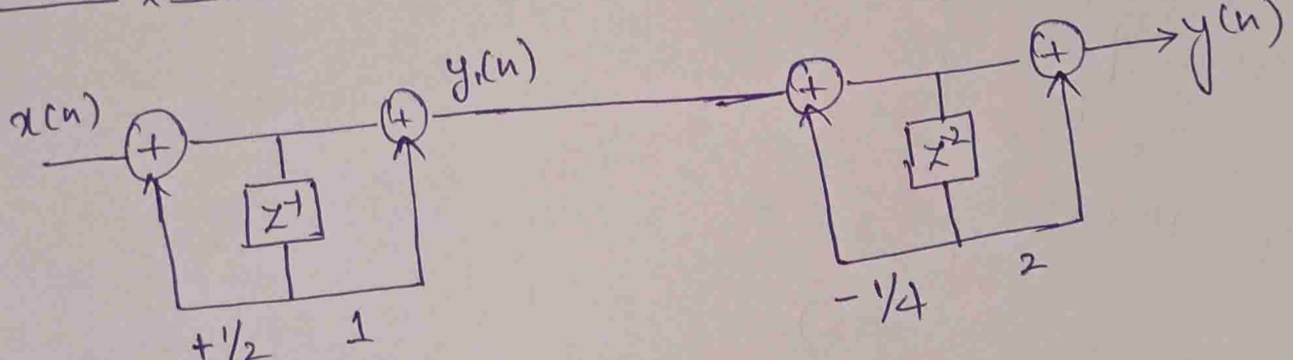
$H(z) = H_1(z) \times H_2(z)$

Direct form I

$H_1(z)$ $H_2(z)$



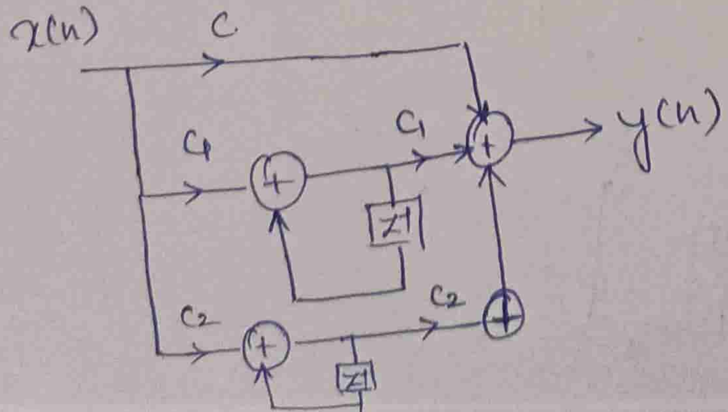
Direct form2



* Parallel form :-

In this form we express $H(z)$ in partial fraction

i.e $H(z) = C + \dfrac{C_1}{1-P_1z^{-1}} + \dfrac{C_2}{1-P_2z^{-1}} + \dots$ → poles.



∴ we get no of parallel connection as many no of partial fraction so many no of parallel connection.

① $y(n) - 3/4\, y(n-1) - 1/8\, y(n-2) + x(n) + \frac{1}{3} x(n-1)$

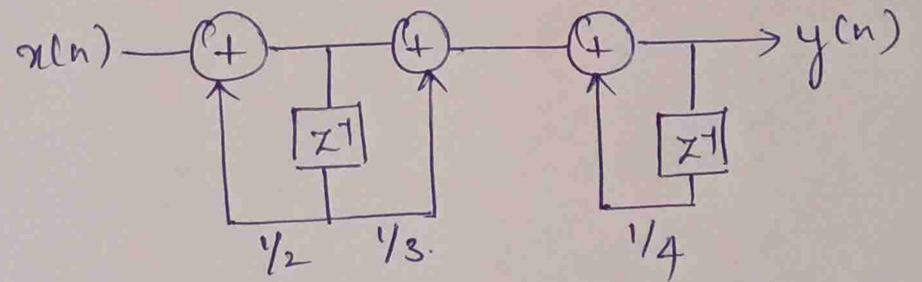
$$y(z) - 3/4\, z^{-1} y(z) + 1/8\, z^{-2} y(z) = x(z) + 1/3\, z^{-1} x(z)$$

$$H(z) = \frac{1 + 1/3\, z^{-1}}{1 - 3/4\, z^{-1} + 1/8\, z^{-2}}$$

$$= \frac{1 + 1/3\, z^{-1}}{(1 - 1/2\, z^{-1})(1 - 1/4\, z^{-1})}$$

$$= \frac{1 + 1/3\, z^{-1}}{(1 - 1/2\, z^{-1})} \times \frac{1}{(1 - 1/4\, z^{-1})}$$

$$= H_1(z) \times H_2(z)$$



Parallel form :-

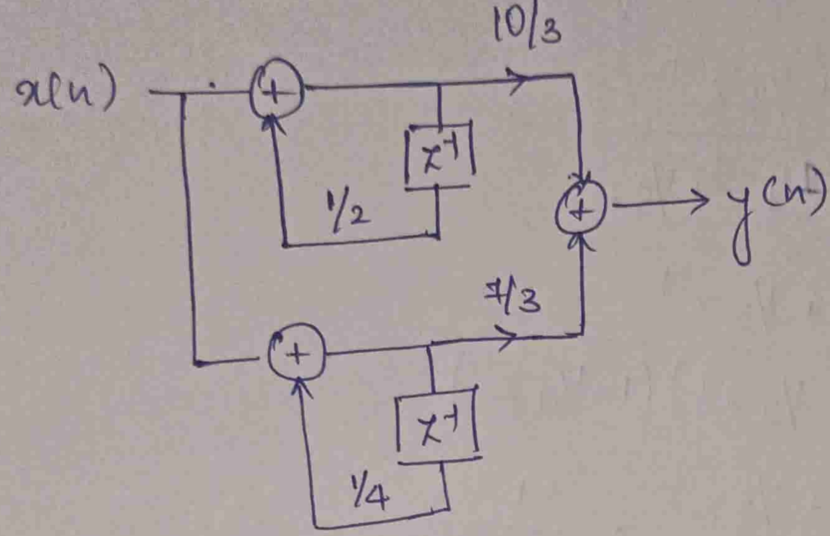
$$H(z) = \frac{A}{(1 - 1/2\, z^{-1})} + \frac{B}{(1 - 1/4\, z^{-1})}$$

Slove for A and B

$$A\Big|_{z^{-1}=2} = \frac{1 + 1/3(z^{-1})}{1 - 1/4(z^{-1})} = \frac{1 + 2/3}{1 - 1/2} = \frac{5/3}{1/2} = 10/3$$

$$B\Big|_{z^{-1}=4} = \frac{1 + 1/3(4)}{1 - 1/2(4)} = \frac{1 + 4/3}{1 - 2} = -7/3$$

$$\therefore H(z) = \frac{10}{3} \cdot \frac{1}{(1-\frac{1}{2}z^{-1})} - \frac{7}{3} \cdot \frac{1}{(1-\frac{1}{4}z^{-1})}$$



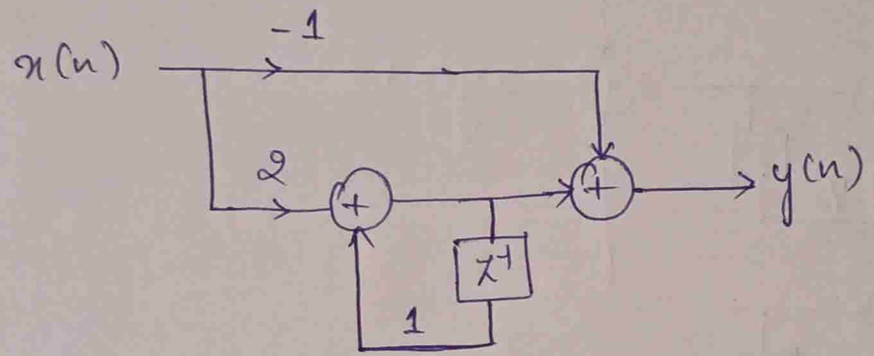
$$H(z) = \frac{1+z^{-1}}{1-z^{-1}}$$

we need to convert it to partial fraction

- numerator and denominator polynomials are same.
  we need to have degree of numerator polynomial less than denominator $\therefore$ divide it by $-z^{-1}+1$.

$$\Rightarrow -z^{-1}+1 \,\big)\, \cancel{z^{-1}}+1 \,\big(-1$$
$$\underline{\cancel{z^{-1}} \underset{(+)}{-} 1}$$
$$2.$$

$$\therefore H(z) = -1 + \frac{2}{1-z^{-1}} = \frac{1+z^{-1}}{1-z^{-1}}$$



2. $$y(z) = \frac{1+3z^{-1}+2z^{-2}}{1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2}}$$

Step 1: degree of denominator is greater.

$$-\tfrac{1}{8}z^{-2}-\tfrac{1}{4}z^{-1}+1 \,\big)\, \cancel{2z^{-2}}+3z^{-1}+1 \,\big(-16$$
$$\underline{-\cancel{2z^{-2}} \underset{-}{+} 4z^{-1} \underset{+}{-} 16}$$
$$-z^{-1}+17$$

$$\therefore y(z) = -16 + \frac{17-z^{-1}}{(1-\frac{1}{4}z^{-1}-\frac{1}{8}z^{-2})}$$

$$= -16 + \frac{17-z^{-1}}{(1-\frac{1}{2}z^{-1})(\frac{1}{4}z^{-1}+1)}$$

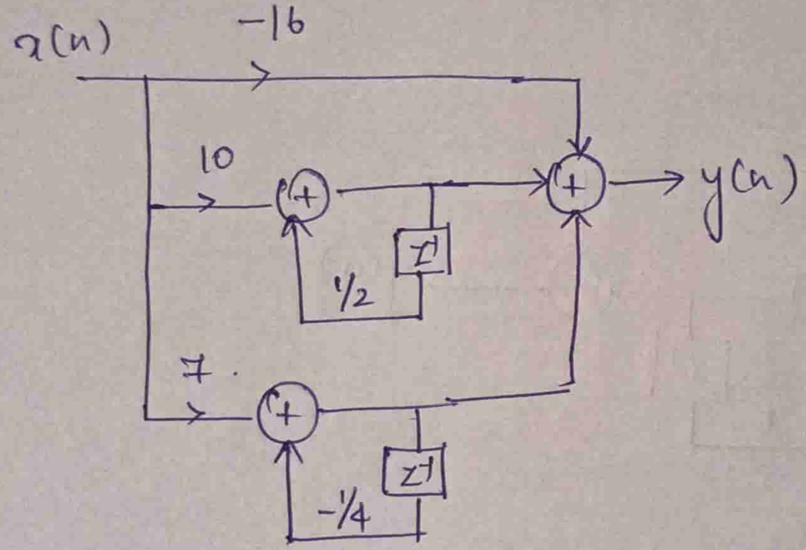
$$= -16 + \frac{A}{1-\frac{1}{2}z^{-1}} + \frac{B}{1+\frac{1}{4}z^{-1}}.$$

To find A :-

$$A\Big|_{z^{-1}=2} = \frac{17-2}{1+\frac{1}{4}(2)} = \frac{15}{3/2} = 10.$$

$$B\Big|_{z=-4} = \frac{17-(-4)}{1-\frac{1}{2}(-4)} = \frac{21}{3} = 7.$$

$$y(z) = -16 + \frac{10}{1-\frac{1}{2}z^{-1}} + \frac{7}{(1+\frac{1}{4}z^{-1})}$$



3. $$H(z) = \frac{1}{1-\frac{5}{6}z^{-1}+\frac{1}{6}z^{-2}} = \frac{1}{(2z^{-1}-1)\,(3z^{-1}-1)}$$

4. $$H(z) = \frac{0.7\,(1-0.36\,z^{-1})}{1-z^{-1}-0.72\,z^{-2}}$$

$$= 0.7\,(z+0.6)$$