Unit - 2.

Time domain Representation of Linear time invariant System

LTI System :- the System which satisfies linearity and time invariant property are called linear - time invariant System.

LTI Systems canbe analysed easily as they exhibit Super position property.

these system canbe represented interms of a linear Combinatio of a set of basic signals and hence we can find the response also interms of basic Signal.

* Various method In which LTI system canbe represented are :-

  1. In terms of impulse response.
  2. Differential equation.
  3. Block diagram. representation

* Impulse response representation for LTI System :-

- A Complete characteristics of any LTI system can be represented in terms of its response to an unit impulse. which is refered as impulse response of a system.

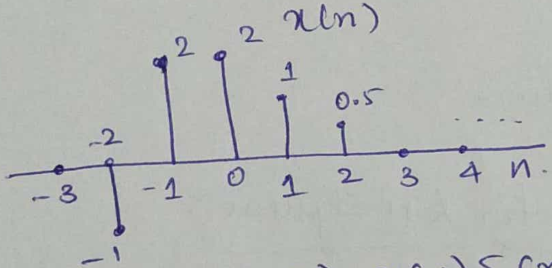
- Impulse response in the o/p of a LTI system is due to the impulse response applied at t=0 or n=0.

If i/p to LTI System is expressed as the weighted Superposition of time shifted impulse, then the o/p is the weighted superposition of system response to each time shifted impulse.

- this weighted Superposition is called Convolution.
- If the System is continuous, then "Convolution integral"
- If the system is discrete then "Convolution Sum".

* Representation for discrete time LTI system in terms of Impulse Response.

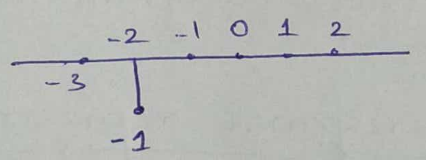
Discuss how to construct any discrete time Signal $x(n)$ in terms of discrete time shifted unit impulse. Lets consider a discrete - time Signal $x(n)$ as shown.



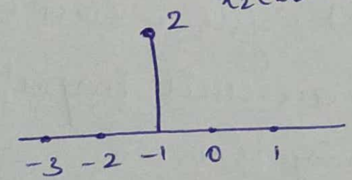
$\Rightarrow$

this signal can be split as shown.

$x_1(n) = x(-2)\,\delta(n+2)$



$x_2(n) = x(-1)\,\delta(n+1)$



$x_3(n) = x(0)\,\delta(n)$

$x_4(n) = x(1)\,\delta(n-1)$

$x_5(n) = x(2)\,\delta(n-2)$

By observing this we can write $x(n)$ as

$$x(n) = x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2)$$

$$= x_1(n) + x_2(n) + x_3(n) + x_4(n) + x_5(n)$$

$$\therefore x(n) = \sum_{k=-2}^{2} x(k)\, \delta(n-k)$$

But in general, any discrete time signal $x(n)$ can be expressed as

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\, \delta(n-k)} \quad -① $$

$\therefore$ We have expressed $x(n)$ as weighted sum of time shifted impulse.

Now Consider a discrete time LTI system

$$\underset{\text{input}}{x(n)} \longrightarrow \underset{\text{System}}{\boxed{T\{\cdot\}}} \longrightarrow \underset{\text{o/p}}{y(n)}$$

System operator.

$$\therefore y(n) = T\{x(n)\} \quad -② $$

Substituting equation ① in ② we get.

$$y(n) = T\left\{ \sum_{n=-\infty}^{\infty} x(k)\, \delta(n-k) \right\}$$

$\therefore$ using linearity property weget.

$$y(n) = \sum_{n=-\infty}^{\infty} x(k) \{\delta(n-k)\} T$$

$\therefore T\{\delta(n-k)\}$ corrosponds to the operation of the systems performed on time shifted impulse $\delta(n-k)$

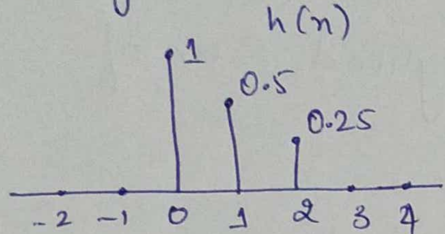
$\therefore T\{\delta(n-k)\} = h(n-k) \rightarrow$ impulse response of the system.

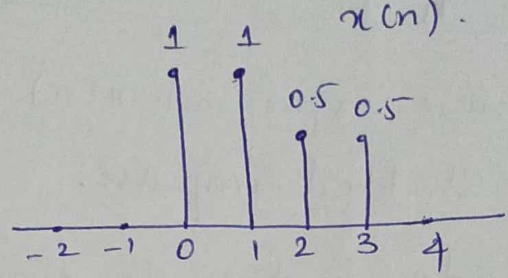
$$\therefore \boxed{y(n) = \sum_{n=-\infty}^{\infty} x(k) \cdot h(n-k)} \Rightarrow \boxed{y(n) = x(n) * h(n)}$$

$\rightarrow$ Convolution Sum.

∴ o/p of LTI System is given by the weighted sum of time-shifted impulse response.

Q. Consider a LTI system having impulse response $h(n)$ and i/p $x(n)$ as shown below obtain its o/p $y(n)$.

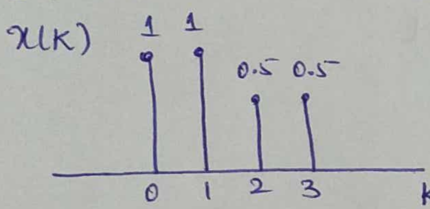




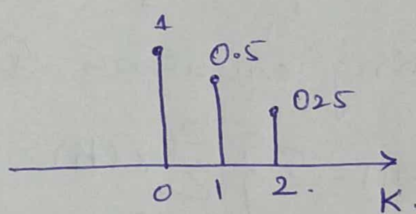
$$y(n) = \sum_{K=-\infty}^{\infty} x(K)\, h(n-K) = \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

because Convolution Satisfies Commutative law and hence o/p will be the same.

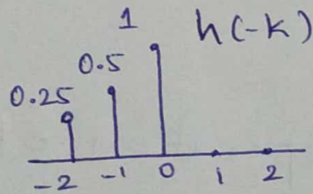
1: obtain $x(K)$ sequence.



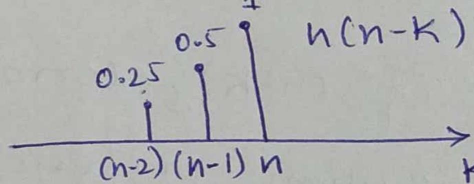
2. Obtain $h(K)$



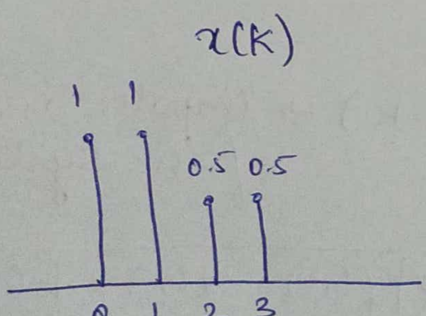
3. flip the sequence to obtain $h(-k)$

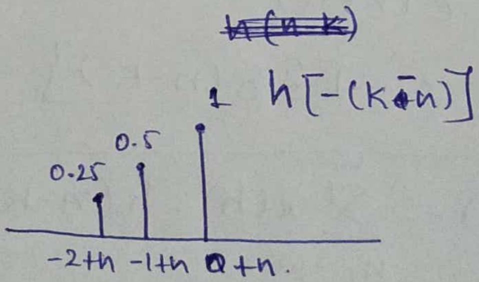


4. Shift the sequence to obtain $h(n-k)$.



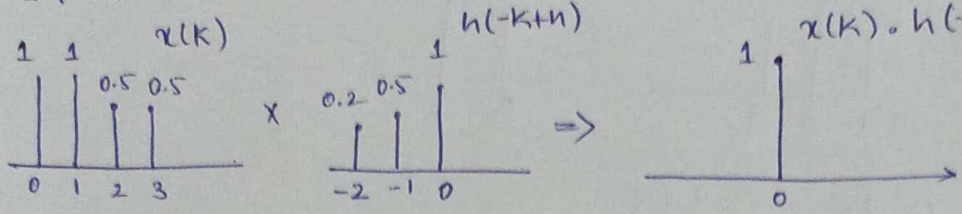
5.





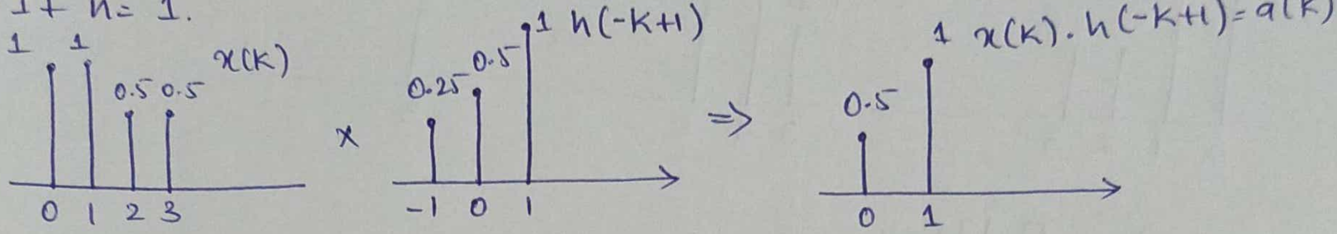
1. If $0+n<0$ then $y(n)=0$

2. If $n=0$



$$y(n) = \sum_{n=0}^{0} x(0).h(0) \Rightarrow y(0) = 0$$

3. If $n=1$.

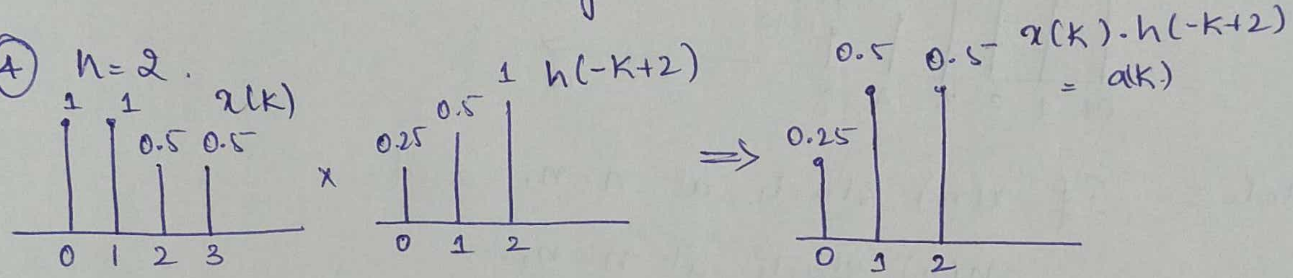


$$y(1) = \sum_{n=0}^{1} x(k)\,h(-k+1) = x(0)\,h(1) + x(1)\,h(0)$$

$$y(1) = a(0) + a(1)$$

$$= 0 + 1 \Rightarrow y(1) = 1.$$

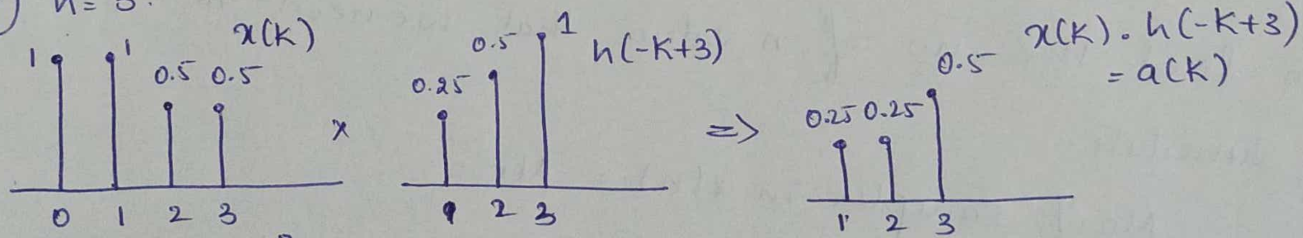
4. $n=2$.



$$y(2) = \sum_{n=0}^{2} a(k) = a(0) + a(1) + a(2)$$

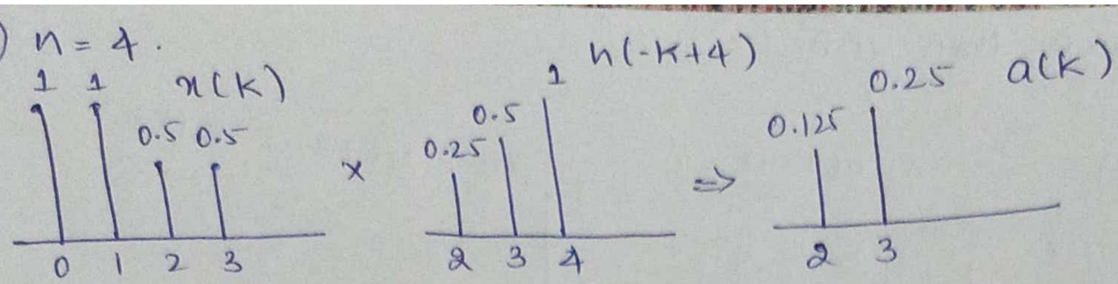
$$= 0.25 + 0.5 + 0.5 = 1.25$$

5. $n=3$.



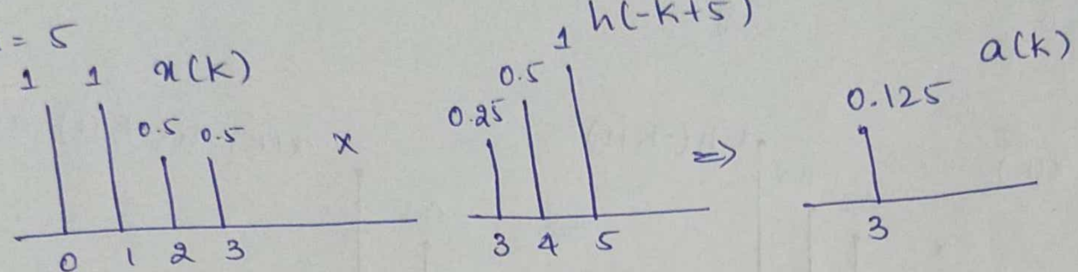
$$y(2) = \sum_{n=01}^{3} a(k) = 0.25 + 0.25 + 0.5 = 1.$$

⑥ $n = 4$.



$y(4) = 0.125 + 0.25 = 0.375$.

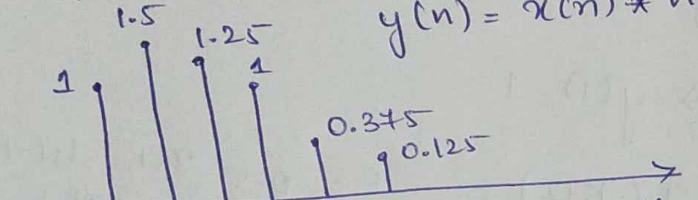
⑦ $n = 5$



$y(5) = 0.125$.

$\therefore y(n) = y(0) + y(1) + y(2) + y(3) + y(4) + y(5)$

$y(n) = x(n) * h(n)$.



Note :- If $x(n)$ starts at $n = n_1$ and

$h(n)$ starts at $n = n_2$

$y(n)$ starts at $n = n_1 + n_2$

$n_1 = 0$ $\quad n_2 = 0$ $\quad \therefore y(n) = 0 + 0 = 0$

To find the no of n values that we need to subsitute.

No of samples in $x(n) = N_1$

No of samples in $h(n) = N_2$

then the no of samples in $y(n)$ is given by $N_1 + N_2 - 1$

$N_1 = 4$ $\quad N_2 = 3$ $\quad N_1 + N_2 - 1 \Rightarrow$ 6 Samples.

$n = (0 \text{ to } 5)$

Q 1:- $x(n) = \{\underset{\uparrow}{1}, 1, 0.5, 0.5\}$ $\quad h(n) = \{\underset{\uparrow}{1}, 0.5, 0.25\}$

$\uparrow$ indicates Origin (zero).

$$y(n) = \sum_{k=0}^{5} x(k) \cdot h(n-k)$$

$$y(0) = x(0) * h(+0) + x(1)\,h(-1) + x(2)\,h(-2) + x(3)\,h(-3) + x(4)\,h(-4) + x(5)\,h(-5)$$

$$y(0) = (1\times 1) + (1\times 0) + (0.5\times 0) + (0.25\times 0)$$

$$y(0) = 1.$$

$$y(1) = \sum_{k=0}^{5} x(k) \cdot h(1-k)$$

$$y(1) = x(0)\,h(1) + x(1)\cdot h(0) + x(2)\,h(-1) + x(3)\,h(-2) + x(4)\,h(-3) + x(5)\,h(-4)$$

$$y(1) = (1\times 0.5) + (1\times 1)$$

$$y(1) = 0.5 + 1 = 1.5$$

$$y(2) = \sum_{k=0}^{5} x(k)\,h(2-k)$$

$$= x(0)\,h(2) + x(1)\,h(1) + x(2)\,h(0) + x(3)\,h(-1)$$

$$= 1\times 0.25 + 1\times 0.5 + 0.5\times 1$$

$$= 0.25 + 0.5 + 0.5$$

$$y(2) = 1.25$$

$$y(3) = \sum_{k=0}^{5} x(k)\,h(3-k)$$

$$= x(0)\,h(3) + x(1)\,h(2) + x(2)\,h(1) + x(3)\,h(0) + x(4)\,h(-1)$$

$$= 1\times 0.25 + 0.5\times 0.5 + 0.5\times 1$$

$$= 0.25 + 0.25 + 0.5 = 1.$$

$$y(4) = \sum_{k=0}^{5} x(k)\, h(4-k)$$

$$= x(0)\,h(4) + x(1)\,h(3) + x(2)\,h(2) + x(3)\,h(1) + x(4)\,h(0)$$

$$= 0.5 \times 0.25 + 0.5 \times 0.5$$

$$= 0.375.$$

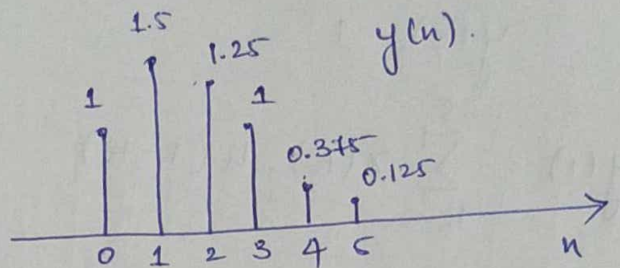
$$y(5) = \sum_{n=0}^{5} x(k)\, h(5-k)$$

$$= x(0)\,h(5) + x(1)\,h(4) + x(2)\,h(3) + x(3)\,h(2) + x(4)\,h(1)$$

$$x(5)\,h(0)$$

$$= 0.5 \times 0.25$$

$$y(5) = 0.125$$



	1	0.5	0.25
1	1	0.5	0.25
1	1	0.5	0.25
0.5	0.5	0.25	0.125
0.5	0.5	0.25	0.125

$$y(n) = \{1, 1.5, 1.25, 1, 0.375, 0.125\}.$$

Representation for Continuous LTI system in terms of impulse response.

Any arbitrary Continuous time signal $x(t)$ can be expressed as the weighted. Superposition of time shifted impulse.

$$x(t) = \int_{-\infty}^{\infty} x(t)\,.\,\delta(t-\tau)\, d\tau \quad -①$$

Consider a LTI System.

$$x(t) \longrightarrow \boxed{S/m} \longrightarrow y(t) \qquad \therefore y(t) = T\{x(t)\}$$

$x(t)$ ↑ i/p to System

$T\{\cdot\}$ ↑ System operator

$y(t)$ ↑ o/p of System.

$$\Rightarrow y(t) = T\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right.$$

By linearity property.

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, T\{\delta(t-\tau)\}$$

$T\{\delta(t-\tau)\} \rightarrow$ operation of the system performed on time shifted impulse. $\delta(t-\tau)$

$$\therefore T\{\delta(t-\tau)\} = h(t-\tau)$$

$$\therefore \boxed{y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau) = x(t) * h(t)}$$

* Properties of ~~impulse~~ convolution

Continuous :-

Discrete.

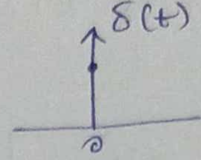
① $x(t) * \delta(t) = x(t)$ $\qquad x(n) * \delta(n) = x(n)$

② $x(t) * \delta(t-t_0) = x(t-t_0)$ $\qquad x(n) * \delta(n-n_0) = x(n-n_0)$

③ $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$ $\qquad x(n) * u(n) = \sum_{k=-\infty}^{n} x(k)$

④ $x(t) * u(t-t_0) = \int_{-\infty}^{t-t_0} x(\tau)\,d\tau$ $\qquad x(n) * u(n-n_0) = \sum_{k=-\infty}^{n-n_0} x(k)$

Proof:-



(1) $$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$$

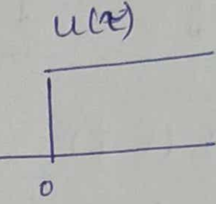
$$= x(\tau)\Big|_{\tau=t}$$

$$= x(t)$$

(2) $$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-t_0-\tau)\, d\tau$$

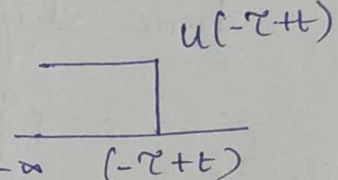
$$= (x(\tau))\Big|_{\tau=t-t_0}$$

$$= x(t-t_0).$$



(3) $$x(t) * u(t) = \int_{-\infty}^{\infty} x(t) \cdot u(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{t} x(\tau) \cdot d\tau$$



(4) $$x(t) * u(t-t_0) = \int_{-\infty}^{\infty} x(t) \cdot u(t-t_0-\tau)\, d\tau$$

$$= \int_{-\infty}^{t-t_0} x(\tau) \cdot d\tau.$$

(5) Commutative property.

Convolution in both Continuous and discrete time are Commutative i.e

$$x(n) * h(n) = h(n) * x(n)$$
$$x(t) * h(t) = h(t) * x(t)$$

$$LHS = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

Let $t-\tau = P \Rightarrow \tau = t-P$ & $d\tau = -dP$

$\tau = -\infty \quad P = \infty$

$\tau = \infty \quad P = -\infty$

$$\therefore \; -\int_{P=\infty}^{-\infty} x(t-P) \cdot h(P) \cdot dP$$

$$\Rightarrow \int_{-\infty}^{\infty} h(P) \cdot x(t-p) \cdot dp = h(t) * x(t)$$

$$\therefore \; x(t) * h(t) = h(t) * x(t)$$

Discrete time signal.

$$x(n) * h(n) = h(n) * x(n)$$

$$LHS = x(n) * h(n) = \sum_{K=-\infty}^{\infty} x(K) \cdot h(n-K)$$

Let $n-K = m \quad \therefore K = n-m$.

$$= \sum_{m=-\infty}^{\infty} x(n-m) \cdot h(m) = h(n) * x(n)$$

$$\therefore \; x(n) * h(n) = h(n) * x(n).$$

(6) <u>Distributive Property</u> :-

$$x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$$

$$x(t) * (h_1(t) + h_2(t)\} = x(t) * h_1(t) + x(t) * h_2(t)$$

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\, d\tau$$

Consider LHS. $= x(t) * [\underbrace{h_1(t) + h_2(t)}_{f(t)}]$

$$\Rightarrow x(t) * f(t) = \int_{-\infty}^{\infty} x(\tau) \cdot f(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) \cdot [h_1(t-\tau) + h_2(t-\tau)]\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, h_1(t-\tau)\, d\tau + \int_{\tau=-\infty}^{\infty} x(\tau)\, h_2(t-\tau)\, d\tau$$

$$= x(t) * h_1(t) + x(t) * h_2(t)$$

$$= \text{RHS.}$$

Hence proved.

$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{f(t)} \boxed{h_2(t)} \longrightarrow y(t)$$

(7) Associative Property:-

$$x(n) * \{h_1(n) * h_2(n)\} = \{x(n) * h_1(n)\} * h_2(n)$$

$$x(t) * \{h_1(t) * h_2(t)\} = \{x(t) * h_1(t)\} * h_2(t)$$

$\therefore$ from figure $y(t) = z(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} z(\tau) \cdot h(t-\tau)\, d\tau \quad - (1)$$

where $z(\tau)$ is the o/p response of first system.

$$\therefore z(t) = x(t) * h_1(t)$$

$$= \int_{-\infty}^{\infty} x(\eta) \cdot h_1(\cancel{t}-\eta)\, d\eta \quad - (2)$$

↳ write in terms of τ replace t by τ)

Substituting eq (2) in (1) we have.

$$y(t) = \int_{\tau-\infty}^{\infty} \int_{-\infty-\eta}^{\infty} x(\eta) \cdot h_1(\tau-\eta) \cdot h_2(t-\tau)\, d\eta\, d\tau$$

Subsitute $\cancel{t}-\eta = m$. and interchange the order of integration

$$y(t) = \int_{\eta=-\infty}^{\infty} x(\eta) \left[\int_{\tau=-\infty}^{\infty} h_1(m) \cdot h_2(t-\eta-m)\, dm \right] d\eta.$$

$$= \int_{\eta=-\infty}^{\infty} x(\eta) \cdot h(\cancel{t}-\eta)\, d\eta$$

where $h(\cancel{t}-\eta) = h_1(t-\eta) * h_2(t-\eta)$

$$= h_1(t) * h_2(t)$$

$\therefore y(t) = x(t) * [h_1(t) * h_2(t)]$ hence proved.

1. $u(t) * \delta(t) = u(t)$

$x(t) * \delta(t) = x(t)$

2. $u(t) * \delta(t-2) = u(t-2)$

$x(t) * \delta(t-t_0) = x(t-t_0)$

3. $u(t-1) * \delta(t-2) = u(t-1-2) = u(t-3)$

4. $u(t+2) * \delta(t+3) = u(t+2+3) = u(t+5)$

5. $u(t-2) * 3\delta(t-3) = 3u(t-2-3)$

$= 3u(t-5)$

6. $2u(t-1) * (-3)\delta(t+3) = -6u(t-1+3)$

$= -6u(t+2)$

7. $\underbrace{t\,u(t)}_{x(t)} * \underbrace{3\delta(t-1)}_{\delta(t-t_0)} = 3t\,u(t-1)$

$x(t) * \delta(t-t_0) = x(\tau)\Big|_{\tau=t-t_0} = x(t-t_0)$

$x(t) * 3\delta(t-1) = 3x(\tau)\Big|_{\tau=t-t_0}$

$= 3 \cdot \tau \cdot u(\tau)\Big|_{\tau=t-1}$

$= 3(t-1)\,u(t-1)$

8. $2t\,u(t-1) * 5\delta(t+2) = 10\tau\, u(\tau-1)\Big|_{\tau=t+2}$

$= 10(t+2)\,u(t-1+2)$

$= (10t+20)\,u(t+1)$

9. $(2t-1)\,u(t-1) * \delta(t+2) = (2\tau-1)\,u(\tau-1)\Big|_{\tau=t+2}$

$= [2(t+2)-1]\,u(t+2-1)$

$= (2t+3)\,u(t+1)$

10. $2\delta(t) * \delta(t-2) = 2\delta(t-2)$

11. given that $x(t) = 2[u(t+1) - u(t-1)]$

$y(t) = \delta(t) - 2\delta(t-1) + 8\delta(t-2)$

find $x(t) * y(t)$

$[2u(t+1) - 2u(t-1)] * [\delta(t) - 2\delta(t-1) + 8\delta(t-2)]$

$= 2u(t+1) - 2u(t-1) - 4u(t+1-1) + 4u(t-1-1) + 16u(t+1-2)$

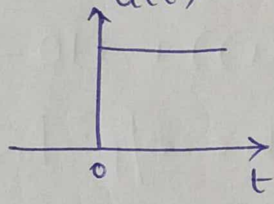
$\quad - 16u(t-1-2)$

$= 2u(t+1) - 2u(t-1) - 4u(t) + 4u(t-2) + 16u(t-1)$

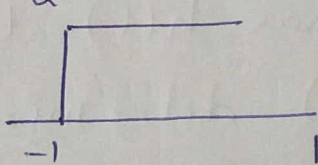
$\quad - 16u(t-3)$

$= 2u(t+1) + 14u(t-1) - 4u(t) + 4u(t-2) - 16u(t-3)$

u(t)



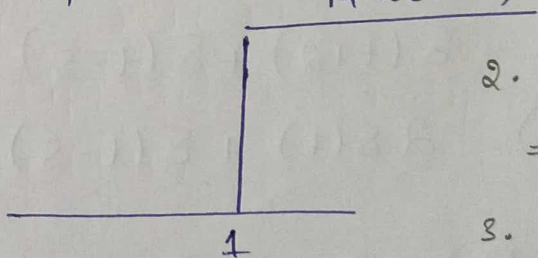
2. $2u(t+1)$



(1) $2 + 14 - 4$

$+14 - 2 = 12$

$14u(t-1)$



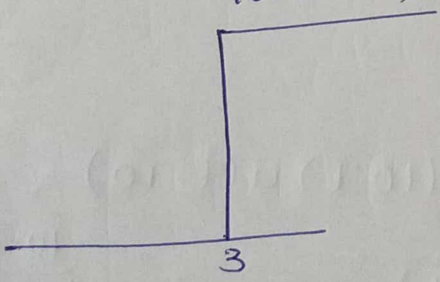
2. $2 + 14 - \cancel{4} + \cancel{4}$

$= 16$

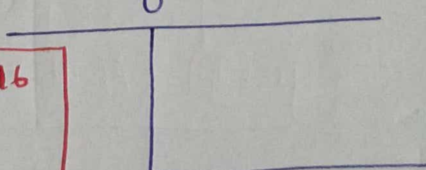
3. $2 + 14 - \cancel{4} + \cancel{4} - 16$

$= 0$

$-16u(t-3)$



0



$-4u(t)$

$-\infty$ to $-1 = 0$

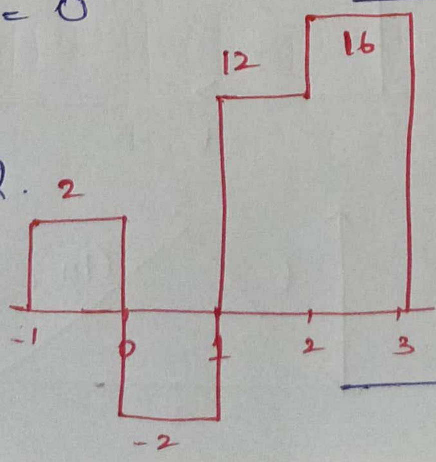
-1 to $0 = 2$.

0 to $1 = -2$.

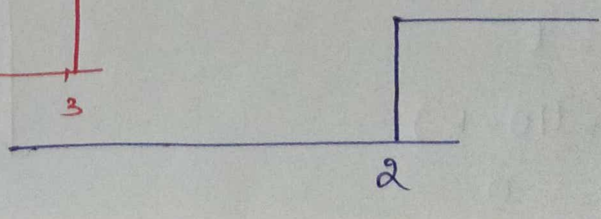
1 to $2 = 12$

2 to $3 = 16$

3 to $\infty = 0$



$+ 4u(t-2)$



(12) $x(t) = 3u(t+2) - u(t)$

$y(t) = 2\delta(t) - 2\delta(t-2)$

$x(t) * y(t)$

$\Rightarrow [3u(t+2) - u(t)] * [2\delta(t) - 2\delta(t-2)]$

$\Rightarrow 6u(t+2) - 2u(t) - 6u(t+2-2) + 2u(t-2)$

$\Rightarrow 6u(t+2) - 8u(t) + 2u(t-2)$

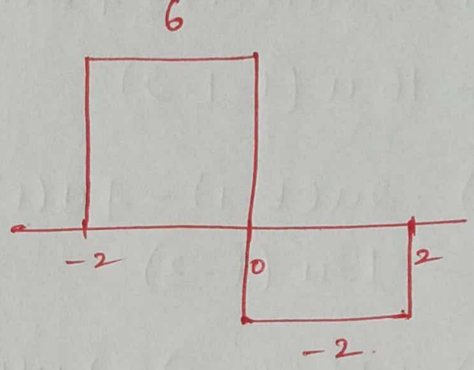
-2 0 +2

$-\infty$ to $-2 \Rightarrow 0$

-2 to $0 \Rightarrow +6$

0 to $1 \Rightarrow -2$

1 to $2 \Rightarrow 0$

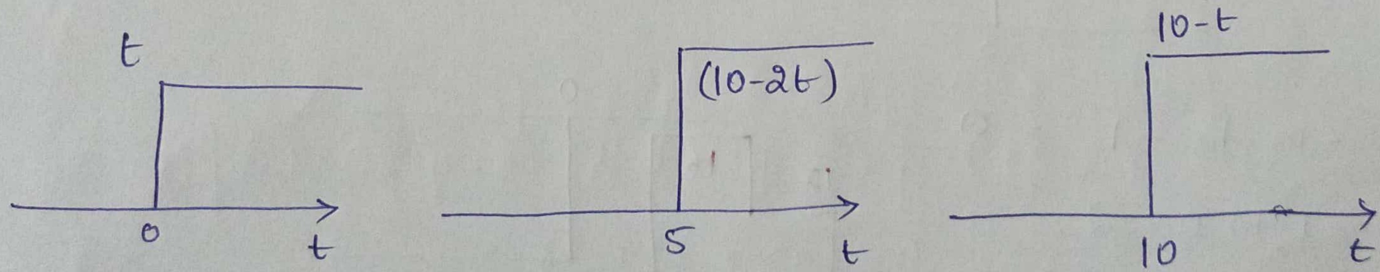


(13) LTI System has $h(t) = t(u(t)) + (10-2t)u(t-5) - (10-t)u(t-10)$ determine the o/p for the i/p

$x_1(t) = \delta(t+2) + \delta(t-5)$

$x_2(t) = 2\delta(t) + \delta(t-5)$

$h(t) = tu(t) + (10-2t)u(t-5) - (10-t)u(t-10)$

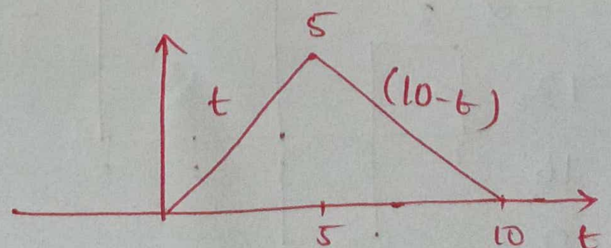


$-\infty$ to $0 = 0$

0 to $5 = t$

5 to $10 = (10-t)$

10 to $\infty = 0$



$y_1(t) = x_1(t) * h(t) = h(t) * x_1(t)$

$y_1(t) = tu(t) + (10-2t)u(t-5) - (10-t)u(t-10) * \delta(t+2) + \delta(t-5)$

$= (t+2)u(t+2) + [10-2(t+2)]\,u(t-5+2) - [10-(t+2)]\,u(t+2-10) +$

$(t-5)\,u(t-5) + [10-2(t-5)]\,u(t-5-5) - [10-(t-5)]\,u(t-5-10)$

$= (t+2)\,\underset{-2}{u(t+2)} + (6-2t)\,\underset{+3}{u(t-3)} - (8-t)\,\underset{+8}{u(t-8)} + (t-5)\,\underset{+5}{u(t-5)}$

$+ (20-2t)\,\underset{+10}{u(t-10)} - (15-t)\,\underset{+15}{u(t-15)}.$

$-\infty$ to $-2 = 0$

-2 to $3 = (t+2)$

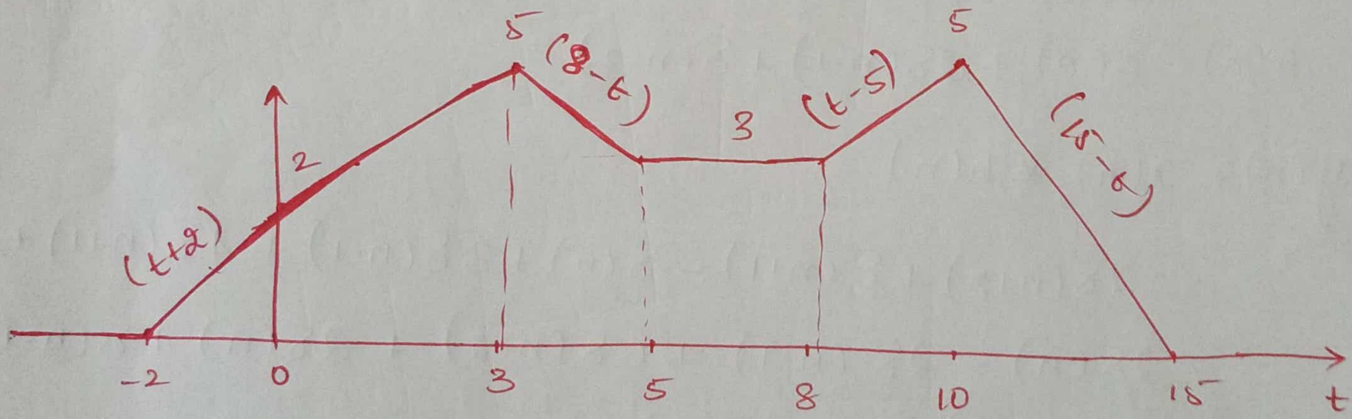
3 to $5 = (t+2) + 6 - 2t\ \cancel{+t-5} = (8-t)$

5 to $8 = (t+2) + 6 - 2t + t - 5 = (8-t) + t - 5 = 3$

8 to $10 = (t+2) + 6 - 2t + t - 5 - 8 + t = (t-5)$

10 to $15 = (t+\cancel{2}) + \cancel{6} - \cancel{2t} + \cancel{t} - 5 - \cancel{8} + \cancel{t} + 20 - 2t = (15-t)$

15 to $\infty = 0$



$y_2(t) = x_2(t) * h(t)$

$= [tu(t) + (10-2t)u(t-5) - (10-t)u(t-10)] * [2\delta(t) + \delta(t-5)]$

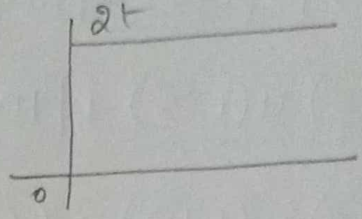
$= 2t\,u(t) + (20-4t)\,u(t-5) - (20-2t)\,u(t-10) + (t-5)\,u(t-5)$

$+ [10-2(t-5)]\,u(t-5-5) - [10-t+5]\,u(t-5-10)$

$= 2t\,u(t) + (20-4t+t-5)\,u(t-5) - (15-t)\,u(t-15)$

$y_2(t) = 2t\,u(t) + (15-3t)\,u(t-5) - (15-t)\,u(t-15)$

$0 \qquad +5 \qquad +15$

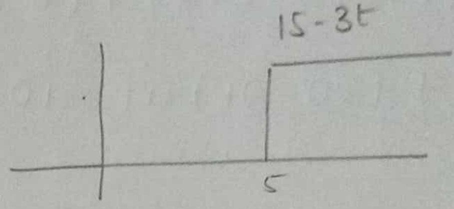


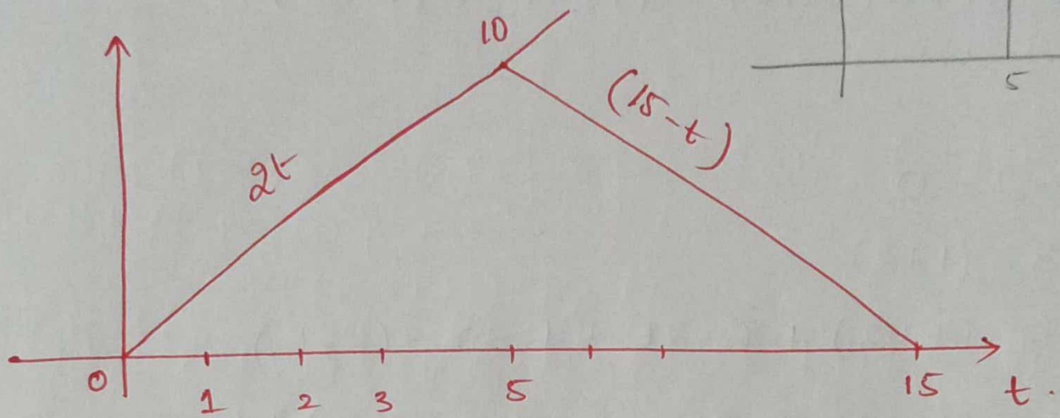
$-\infty$ to $0 = 0$

0 to $5 = 2t$

5 to $15 = 2t + 15 - 3t = 15 - t$

15 to $\infty = 0$





14. Compute $y(n) = x(n) * h(n)$ where $x(n) = \{2, 1, \underset{\uparrow}{-1}, 3\}$

$h(n) = \{\underset{\uparrow}{1}, 2, 1\}$

$x(n) = 2\delta(n+2) + \delta(n+1) - \delta(n) + 3\delta(n-1)$

$h(n) = \delta(n) + 2\delta(n-1) + \delta(n-2)$

$y(n) = x(n) * h(n)$

$= 2\delta(n+2) + \delta(n+1) - \delta(n) + 3\delta(n-1) + 4\delta(n+1) + 2\delta(n) - 2\delta(n-1) + 6\delta(n-2) + 2\delta(n) + \delta(n-1) - \delta(n-2) + 3\delta(n-3)$

$= 2\delta(n+2) + 5\delta(n+1) + 3\delta(n) + 2\delta(n-1) + 5\delta(n-2) + 3\delta(n-3)$

$y(n) = \{2, 5, \underset{\uparrow}{+3}, 2, 5, 3\}$