Signals and Systems UNIT 5

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Introduction to Z transforms

- A generalization of Fourier transform
- Why generalize it?
  - FT does not converge on all sequence
  - Notation good for analysis
  - Bring the power of complex variable theory deal with the discrete-time signals and systems

Relation between Z transforms and DTFT

- The z-transform of sequence $x(n)$ is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- Let $z = e^{-j\omega}$.

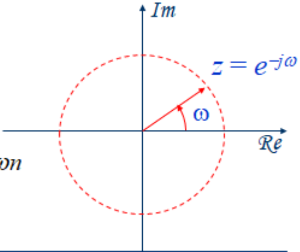
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

Fourier Transform

Z plane

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$



Fourier Transform is to *evaluate z-transform on a unit circle.*

Region of Convergence (ROC)

- Give a sequence, the set of values of z for which the z-transform converges, i.e., $|X(z)|<\infty$, is called the region of convergence.

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n) z^{-n} \right| = \sum_{n=-\infty}^{\infty} |x(n)| |z|^{-n} < \infty$$

ROC is centered on origin and consists of a set of rings.

Example: Region of Convergence (ROC)

EXAMPLE 7.1 Determine the z-transform of the signal

$$x[n] = \begin{cases} 1, & n = -1 \\ 2, & n = 0 \\ -1, & n = 1 \\ 1, & n = 2 \\ 0, & \text{otherwise} \end{cases}$$

Solution: Substitute $x[n]$ into Eq. (7.4) to obtain

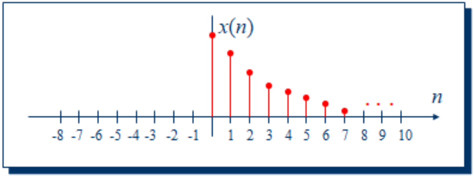
$$X(z) = z + 2 - z^{-1} + z^{-2}$$

We obtain the DTFT from $X(z)$ by substituting $z = e^{j\Omega}$:

$$X(e^{j\Omega}) = e^{j\Omega} + 2 - e^{-j\Omega} + e^{-j2\Omega}$$

Example: A right sided Sequence

$$x(n) = a^n u(n)$$



Example: Z transform of A right sided Sequence

$$x(n) = a^n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |az^{-1}| < \infty \Rightarrow |az^{-1}| < 1$$

$$\Rightarrow |z| > |a|$$

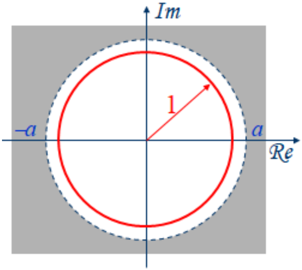
$$X(z) = \sum_{n=0}^{\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

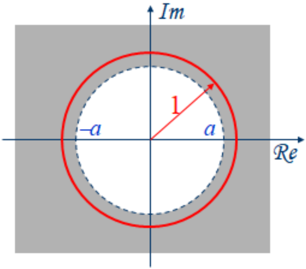
$$|z| > |a|$$

ROC

$$X(z) = \frac{z}{z-a}, \qquad |z| > |a|$$

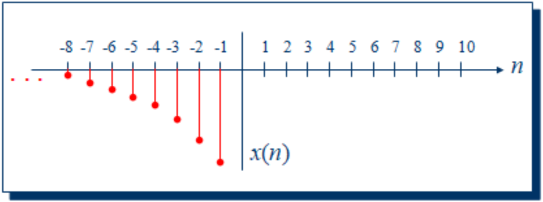
Which one is stable?





Example: A Left sided Sequence

$$x(n) = -a^n u(-n-1)$$



Example: Z transform of A Left sided Sequence

$$x(n) = -a^n u(-n-1)$$

$$X(z) = -\sum_{n=-\infty}^{\infty} a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n}$$

$$= -\sum_{n=1}^{\infty} a^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} a^{-n} z^n$$

For convergence of $X(z)$, we require that

$$\sum_{n=0}^{\infty} |a^{-1} z| < \infty \Rightarrow |a^{-1} z| < 1$$

$$\Rightarrow |z| < |a|$$

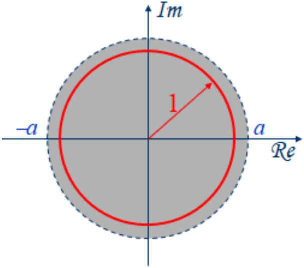
$$X(z) = 1 - \sum_{n=0}^{\infty} (a^{-1} z)^n = 1 - \frac{1}{1 - a^{-1} z} = \frac{z}{z - a}$$

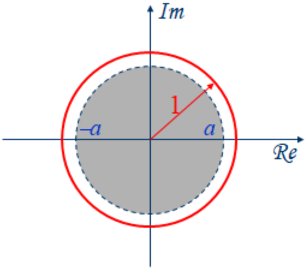
$$|z| < |a|$$

ROC

$$X(z) = \frac{z}{z-a}, \qquad |z| < |a|$$

Which one is stable?





Poles and Zeros of Zplane

$$X(z) = \frac{P(z)}{Q(z)}$$

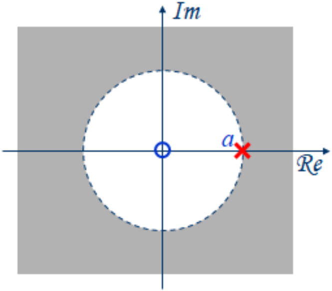
where $P(z)$ and $Q(z)$ are polynomials in z.

Zeros: The values of z's such that $X(z) = 0$

Poles: The values of z's such that $X(z) = \infty$

Poles and ROC of A right sided Sequence

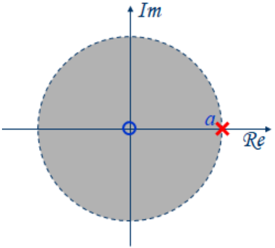
$$x(n) = a^n u(n) \Rightarrow X(z) = \frac{z}{z-a}, \quad |z| > |a|$$



ROC is bounded by the pole and is the exterior of a circle.

Poles and ROC of A Left sided Sequence

$$x(n) = -a^n u(-n-1) \Rightarrow X(z) = \frac{z}{z-a}, \quad |z| < |a|$$

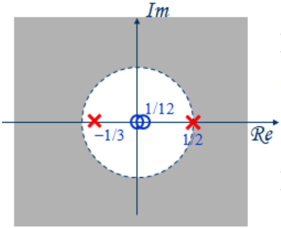


ROC is bounded by the pole and is the interior of a circle.

Poles and ROC of Sum of Two Right Sided Sequences

$$x(n) = (\tfrac{1}{2})^n u(n) + (-\tfrac{1}{3})^n u(n)$$

$$X(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{2z(z-\frac{1}{12})}{(z-\frac{1}{2})(z+\frac{1}{3})}$$



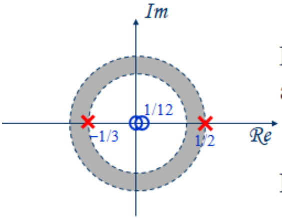
ROC is bounded by poles and is the exterior of a circle.

ROC does not include any pole.

Poles and ROC of Two Sided Sequences

$$x(n) = (-\tfrac{1}{3})^n u(n) - (\tfrac{1}{2})^n u(-n-1)$$

$$X(z) = \frac{z}{z+\frac{1}{3}} + \frac{z}{z-\frac{1}{2}} = \frac{2z(z-\frac{1}{12})}{(z+\frac{1}{3})(z-\frac{1}{2})}$$



ROC is bounded by poles and is a ring.

ROC does not include any pole.

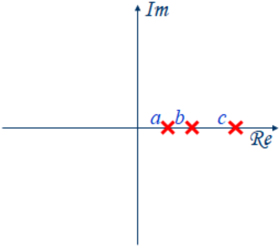
Properties of ROC

- A ring or disk in the z-plane centered at the origin.
- The Fourier Transform of $x(n)$ is converge absolutely iff the ROC includes the unit circle.
- The ROC cannot include any poles
- Finite Duration Sequences: The ROC is the entire z-plane except possibly z=0 or z=∞.
- Right sided sequences: The ROC extends outward from the outermost finite pole in $X(z)$ to z=∞.
- Left sided sequences: The ROC extends inward from the innermost nonzero pole in $X(z)$ to z=0.

Exercise to decide ROC

Consider the rational z-transform with the pole pattern:

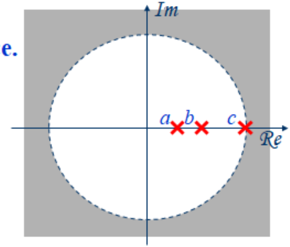
Find the possible ROC's



Case 1

Consider the rational z-transform with the pole pattern:

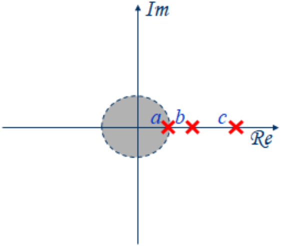
Case 1: A right sided Sequence.



Case 2

Consider the rational z-transform with the pole pattern:

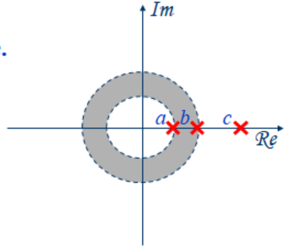
Case 2: A left sided Sequence.



Case 3

Consider the rational z-transform with the pole pattern:

Case 3: A two sided Sequence.



Linearity Property

$$
\begin{aligned}
x_1(n) &\overset{z}{\longleftrightarrow} X_1(z), \text{ROC}_1 \\
x_2(n) &\overset{z}{\longleftrightarrow} X_2(z), \text{ROC}_2
\end{aligned}
$$

$$
\begin{aligned}
a_1x_1(n) + a_2x_2(n) &\overset{z}{\longleftrightarrow} a_1X_1(z) + a_2X_2(z), \\
&\quad \text{At least } \text{ROC}_1 \cap \text{ROC}_2
\end{aligned}
$$

Linearity Property Proof

Let $x(n) = a_1x_1(n) + a_2x_2(n)$

$$\begin{aligned} X(z) &= \sum_{n=-\infty}^{\infty} x(n)z^{-n} \\ &= \sum_{n=-\infty}^{\infty} (a_1x_1(n) + a_2x_2(n))z^{-n} \\ &= a_1 \sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + a_2 \sum_{n=-\infty}^{\infty} x_2(n)z^{-n} \\ &= a_1X_1(z) + a_2X_2(z) \end{aligned}$$

ROC: At least $ROC_1 \cap ROC_2$

Time Shifting

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC}$$

$$x(n-k) \overset{z}{\longleftrightarrow} z^{-k}X(z), \qquad \text{At least ROC}$$

except $z = 0$ $(k > 0)$ or
$z = \infty$ $(k < 0)$

Time Shifting Proof

Let $y(n) = x(n-k)$.

$$\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} (x(n-k)) z^{-n} \\
&\text{Let } m = n-k \\
&= \sum_{m=-\infty}^{\infty} x(m) z^{-(m+k)} = \sum_{m=-\infty}^{\infty} x(m) z^{-m} z^{-k} \\
&= z^{-k} \left[\sum_{m=-\infty}^{\infty} x(m) z^{-m} \right] = z^{-k} X(z)
\end{aligned}$$

Scaling in the Z domain

$$x(n) \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC: } r_1 < |z| < r_2$$

$$a^n x(n) \overset{z}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC: } |a|r_1 < |z| < |a|r_2$$

Scaling in the Z domain Proof

Let $y(n) = a^n x(n)$

$$\begin{aligned} Y(z) &= \sum_{n=-\infty}^{\infty} y(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n} \\ &= \sum_{n=-\infty}^{\infty} x(n)(a^{-1}z)^{-n} \\ &= X(a^{-1}z) \end{aligned}$$

ROC: $r_1 < |a^{-1}z| = \frac{|z|}{|a|} < r_2 \equiv |a|r_1 < |z| < |a|r_2$

Convolution Property

$$x(n) = x_1(n) * x_2(n) \iff X(z) = X_1(z) \cdot X_2(z)$$

Convolution Property Proof

$$x_1(n) * x_2(n) \longleftrightarrow X_1(Z).X_2(Z)$$

Proof –

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n)Z^{-n}$$

$$= \sum_{n=-\infty}^{\infty}[\sum_{k=-\infty}^{\infty} x_1(k)x_2(n-k)]Z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n}^{\infty} x_2(n-k)Z^{-n}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k)[\sum_{n=-\infty}^{\infty} x_2(n-k)Z^{-(n-k)}Z^{-k}]$$

Convolution Property Proof (Conti...)

Let n-k = l, then the above equation cab be written as –

$$X(Z) = \sum_{k=-\infty}^{\infty} x_1(k)[Z^{-k} \sum_{l=-\infty}^{\infty} x_2(l) Z^{-l}]$$

$$= \sum_{k=-\infty}^{\infty} x_1(k) X_2(Z) Z^{-k}$$

$$= X_2(Z) \sum_{k=-\infty}^{\infty} x_1(Z) Z^{-k}$$

$$= X_1(Z).X_2(Z)$$

$$Hence Proved$$

Differentiation Property

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $nx(n) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$ with ROC = R

Proof:

z transform is given by

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Differentiation Property Proof

Differentiating above on both sides with respect to 'z'

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\} = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}\{z^{-n}\} = \sum_{n=-\infty}^{\infty} -nx(n)z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx(n)z^{-n}$$

Comparing both equations $-z\frac{dX(z)}{dz}$ is the z transform of $nx(n)$.

ROC remains the same R because differentiating X(z) will increase the order of the poles present at the same location as earlier.

The End