# Signals and Systems UNIT 3

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# LTI system Properties in terms of impulse response

Please follow the below link for study material. Link

# Step Response

Characterizes the response to sudden changes in input.

$$s[n] = h[n] * u[n]$$

$$= \sum_{k=-\infty}^{\infty} h[k]u[n-k]$$

u[n-k] exist from $-\infty$ to n

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

h[k] is the running sum of impulse response.

# Step Response for Continuous time system

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$$

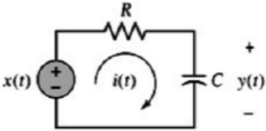
Relation b/w step response and impulse response.

$$h(t) = \frac{d}{dt} s(t)$$

$$h[n] = s[n] - s[n-1]$$

Eg. Step response of an RC ckt

$$h(t) = \frac{1}{RC} e^{-\frac{t}{RC}} u(t)$$



# Step Response Example

Find the step response for LTI system represented by impulse response $h[n] = (\frac{1}{2})^n u[n]$.

Ans:

$$s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$s[n] = \sum_{k=0}^{n} (\frac{1}{2})^k$$

$$= \frac{1-(\frac{1}{2})^{n+1}}{1-\frac{1}{2}}, n \geq 0$$

# Step Response Example

$$h(t) = tu(t)$$

Ans:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(t) = \int_{0}^{t} \tau d\tau$$

$$= \frac{t^2}{2}, t \geq 0$$

# Differential and Difference equation representation of LTI systems

Linear constant coefficient differential equation:

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Linear constant coefficient difference equation:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Order of the equation is (N,M), representing the number of energy storage devices in the system.

Often N>M and the order is described using only 'N'.

# Differential and Difference equation representation of LTI systems

Second order difference equation:

$$y[n] + y[n-1] + \frac{1}{4}y[n-2] = x[n] + 2x[n-1]$$

Difference equation are easily arranged to obtain recursive formulas for computing the current o/p of the system.

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k] - \frac{1}{a_0}\sum_{k=1}^{N} a_k y[n-k] \qquad (1)$$

(1) Shows how to obtain y[n] from present and past values of the input.

# Recursive evaluation of difference equation

Find the first two o/p values y[0] and y[1] for the system described by $y[n] = x[n] + 2x[n-1] - y[n-1] - \frac{1}{4}y[n-2]$.

Assuming that the input is $x[n] = (\frac{1}{2})^n u[n]$ and the initial conditions are y[-1]=1 and y[-2]=-2.

$$y[0] = x[0] + 2x[-1] - y[-1] - \frac{1}{4}y[-2]$$

$$y[1] = x[1] + 2x[0] - y[0] - \frac{1}{4}y[-1]$$

$$y[0] = 1 + 2\times 0 - 1 - \frac{1}{4}\times(-2) = \frac{1}{2}$$

$$y[1] = \frac{1}{2} + 2\times 1 - \frac{1}{2} - \frac{1}{4}\times(1) = 1\frac{3}{4}$$

# Solving Differential and Difference equations

Output of LTI system described by differential or difference equation has two components

$y^h$ - homogeneous solution

$y^p$ - particular solution

Complete solution

$$y = y^h + y^p$$

Eg.

A system is described y the difference equation

$$y[n]-1.143y[n-1]+0.4128y[n-2]=0.0675x[n]+0.1349x[n-1]+0.675x[n-2]$$

$y[-1]=1$ and $y[-2]=2$,

# Homogeneous Solution for CT system

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = \sum_{k=0}^{M} b_k \frac{d^k}{dt^k} x(t)$$

Set all terms involving input to zero

$$\sum_{k=0}^{N} a_k \frac{d^k}{dt^k} y(t) = 0$$

Solution

$$y^h(t) = \sum_{i=1}^{N} c_i e^{r_i t}$$

$r_i$ are the N roots of the *characteristic equation*

# Homogeneous Solution for DT system

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Set all terms involving input to zero

$$\sum_{k=0}^{N} a_k y[n-k] = 0$$

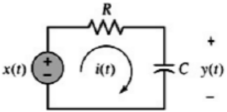
Solution

$$y^h[n] = \sum_{i=1}^{N} c_i {r_i}^n$$

$r_i$ are the N roots of the characteristic equation.

# Homogeneous Solution Example

RC ckt depicted in figure is described by the differential equation $y(t) + RC\frac{d}{dt}y(t) = x(t)$ . Determine the homogenous solution.




The homogenous equation is

$$y(t) + RC\frac{d}{dt}y(t) = 0 \qquad (1)$$

Solution

$$y^h(t) = \sum_{i=1}^{N} c_i e^{r_i t}$$

$$y^h(t) = c_1 e^{r_1 t}$$

Characteristic equation

$$(1 + RCr_1) = 0$$

$$r_1 = -\frac{1}{RC}$$

Homogenous solution of the system is

$$y^h(t) = c_1 e^{-\frac{1}{RC}t}$$

# Homogeneous Solution Example

Determine the homogenous solution.

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 2x(t) + \frac{d}{dt}x(t)$$

Ans:

Homogenous equation

$$\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = 0 \quad (1)$$

Homogenous solution

$$y^h(t) = \sum_{i=1}^{N} c_i e^{r_i t}$$

$$y^h(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

To find $r_1$ and $r_2$ solve CE.

Put $\frac{d^n}{dt^n}y(t) = r^n$

(1) becomes

$$r^2 + 5r + 6 = 0$$

$$r = -3, -2$$

$$y^h(t) = c_1 e^{-3t} + c_2 e^{-2t}$$

# Homogeneous Solution Example

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=x[n-1]$$

**Ans:**

Set input to zero

$$y[n]-\frac{1}{4}y[n-1]-\frac{1}{8}y[n-2]=0 \quad (1)$$

N=2

$$y^h[n]=c_1{r_1}^n+c_2{r_2}^n$$

(1) Becomes

$$1-\frac{1}{4}r^{-1}-\frac{1}{8}r^{-2}=0$$

$$r^2-\frac{1}{4}r-\frac{1}{8}=0$$

$$r=\frac{1}{2},-\frac{1}{4}$$

$$y^h[n]=c_1(\frac{1}{2})^n+c_2(-\frac{1}{4})^n$$

# The Particular Solution

Solution of difference or differential equation for a given input.

Assumption : output is of same general form as the input.

| CT System | |
|---|---|
| Input | Particular solution |
| 1 | $c$ |
| t | $c_1 t + c_2$ |
| $e^{-at}$ | $ce^{-at}$ |
| $\cos(\omega t + \Phi)$ | $c_1 \cos(\omega t) + c_2 \sin(\omega t)$ |

| DT System | |
|---|---|
| Input | Particular solution |
| 1 | $c$ |
| n | $c_1 n + c_2$ |
| $\alpha^n$ | $c\alpha^n$ |
| $\cos(\Omega n + \Phi)$ | $c_1 \cos(\Omega n) + c_2 \sin(\Omega n)$ |

# The Complete Solution

Procedure for calculating complete solution

1. Find the form of the homogeneous solution $y^h$ from the roots of the CE equation.
2. Find a particular solution $y^p$ by assuming that it is of the same form as the input, yet is independent of all terms in the homogeneous solution.
3. Determine the coefficients in the homogeneous solution so that the complete solution $y = y^h + y^p$ satisfies the initial conditions.

# The Complete Solution Example

Find the solution for the first order recursive system described by the difference equation.

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

If the input $x[n] = (\frac{1}{2})^n u[n]$ and the initial condition is $y[-1] = 8$.

**Ans:**

N=1

Homogenous solution

$$y[n] - \frac{1}{4}y[n-1] = 0$$

$$y^h[n] = c_1 r^n$$

$$r^n - \frac{1}{4}r^{n-1} = 0$$

$$r - \frac{1}{4} = 0$$

$$y^h[n] = c_1(\frac{1}{4})^n$$

Particular solution

$$y^p(n) = c_2(\frac{1}{2})^n$$

$$c_2(\frac{1}{2})^n - \frac{1}{4}c_2(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$$

$$c_2 - \frac{1}{2}c_2 = 1$$

$$c_2 = 2$$

# The Complete Solution Example Conti...



Find the solution for the first order recursive system described by the difference equation.

$$y[n] - \frac{1}{4}y[n-1] = x[n]$$

If the input $x[n] = (\frac{1}{2})^n u[n]$ and the initial condition is $y[-1] = 8$.

**Ans:**

N=1

Homogenous solution

$$y[n] - \frac{1}{4}y[n-1] = 0$$

$$y^h[n] = c_1 r^n$$

$$r^n - \frac{1}{4}r^{n-1} = 0$$

$$r - \frac{1}{4} = 0$$

$$y^h[n] = c_1(\frac{1}{4})^n$$

Particular solution

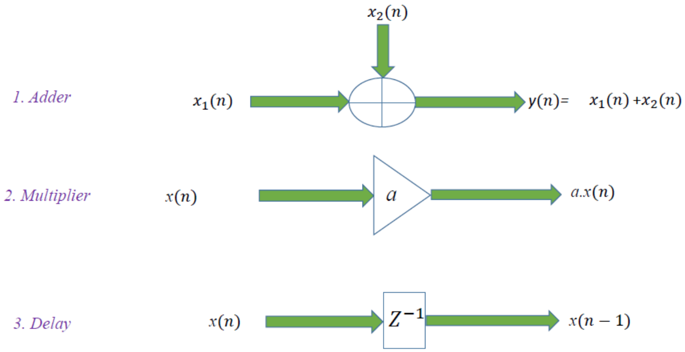
$$y^p(n) = c_2(\frac{1}{2})^n$$

$$c_2(\frac{1}{2})^n - \frac{1}{4}c_2(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$$

$$c_2 - \frac{1}{2}c_2 = 1$$

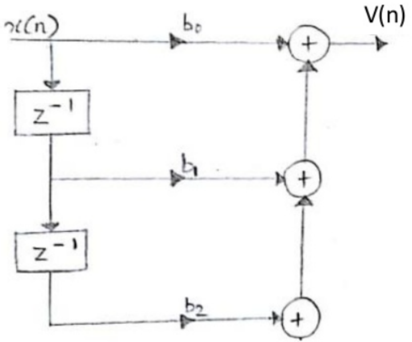
$$c_2 = 2$$

# Basic Structures



# System I Implementation using Basic Structures

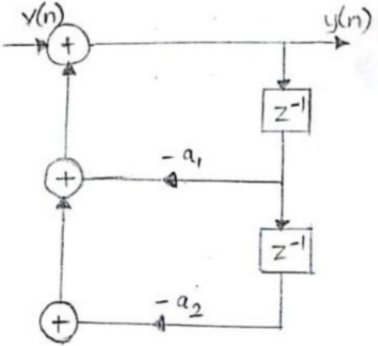
$$v(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$




System-I

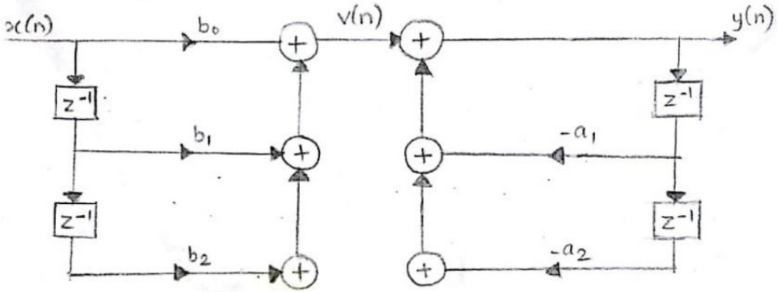
# System II Implementation using Basic Structures

$$y(n) = v(n) - a_1 y(n-1) - a_2 y(n-2)$$



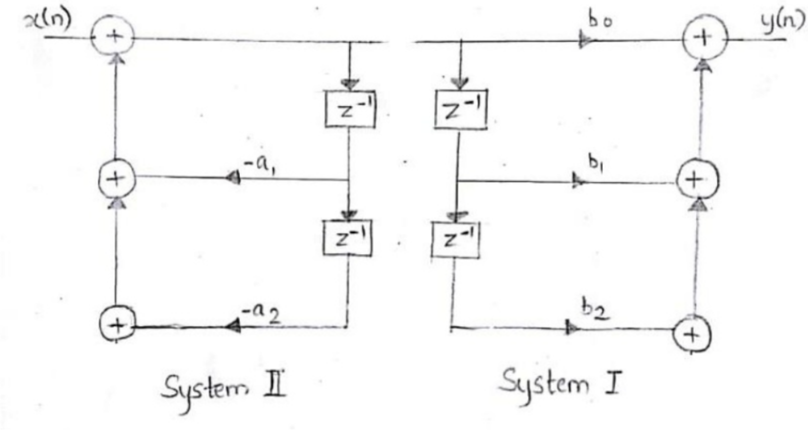

System-II

# Combined System Implementation

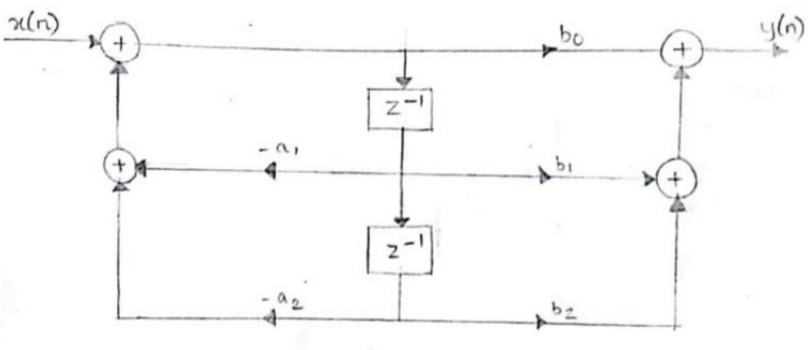



$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

# Direct Form II



# Direct Form II Realization



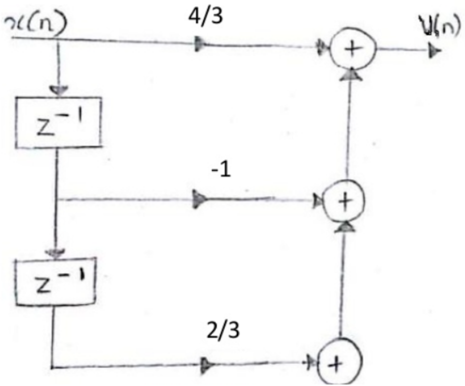
# Example

**Implement the system represented by following difference euqation**

$$y(n) = \frac{4}{3}x(n) - x(n-1) + \frac{2}{3}x(n-2) + \frac{2}{3}y(n-1) - \frac{1}{3}y(n-2)$$

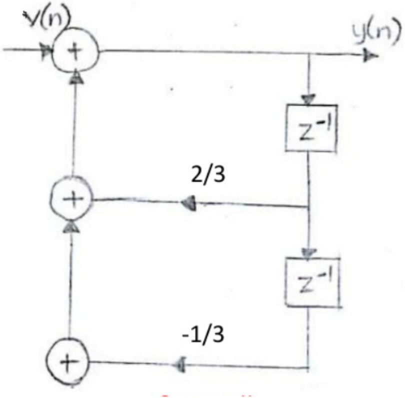
# System 1 Implementation

$$v(n) = \frac{4}{3}x(n) - x(n-1) + \frac{2}{3}x(n-2)$$

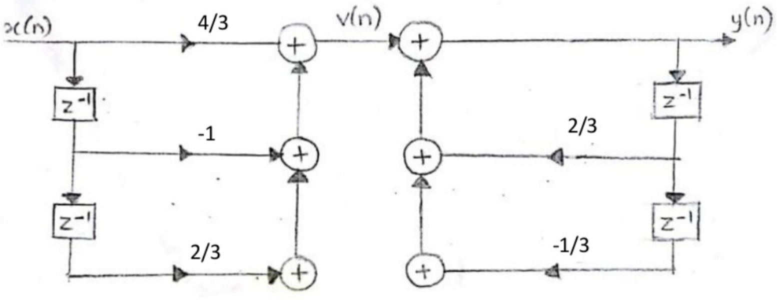


# System 2 Implementation

$$y(n) = v(n) + \frac{2}{3}y(n-1) - \frac{1}{3}y(n-2)$$



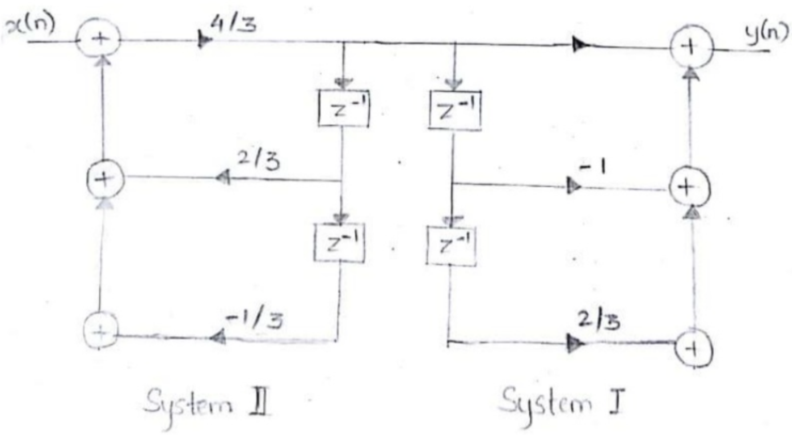
# Direct Form 1 Realization



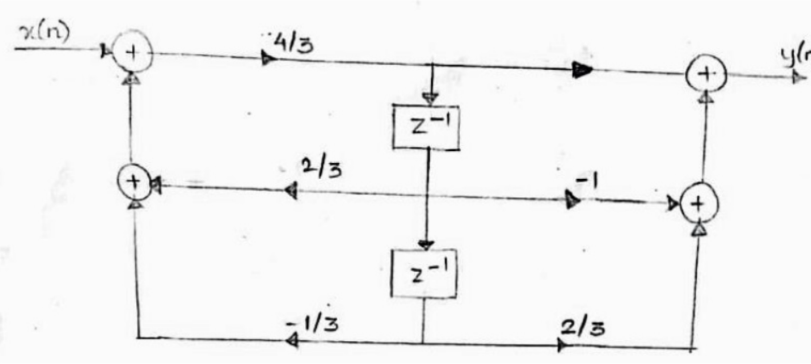

System-I

System-II

# Changing the location of $z^{-1}$*structures*



# Direct Form 1 Realization



# The End