# Control Systems UNIT 5
# State Space Analysis

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# State Space Analysis

The **state space model** of Linear Time-Invariant (LTI) system can be represented as,

$$\dot{X} = AX + BU$$

$$Y = CX + DU$$

The first and the second equations are known as state equation and output equation respectively.

Where,

- X and $\dot{X}$ are the state vector and the differential state vector respectively.
- U and Y are input vector and output vector respectively.
- A is the system matrix.
- B and C are the input and the output matrices.
- D is the feed-forward matrix.

# Basic Concepts of State Space Model

## State

It is a group of variables, which summarizes the history of the system in order to predict the future values (outputs).

## State Variable

The number of the state variables required is equal to the number of the storage elements present in the system.

**Examples** – current flowing through inductor, voltage across capacitor

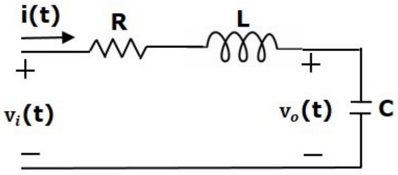
## State Vector

It is a vector, which contains the state variables as elements.

In the earlier chapters, we have discussed two mathematical models of the control systems. Those are the differential equation model and the transfer function model. The state space model can be obtained from any one of these two mathematical models. Let us now discuss these two methods one by one.

# Example 1

Consider the following series of the RLC circuit. It is having an input voltage, $v_i(t)$ and the current flowing through the circuit is $i(t)$ .




There are two storage elements (inductor and capacitor) in this circuit. So, the number of the state variables is equal to two and these state variables are the current flowing through the inductor, $i(t)$ and the voltage across capacitor, $v_c(t)$ .

# Solution Example 1

From the circuit, the output voltage, $v_0(t)$ is equal to the voltage across capacitor, $v_c(t)$ .

$$v_0(t) = v_c(t)$$

Apply KVL around the loop.

$$v_i(t) = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$$

$$\Rightarrow \frac{di(t)}{dt} = -\frac{Ri(t)}{L} - \frac{v_c(t)}{L} + \frac{v_i(t)}{L}$$

# Solution Example 1

The voltage across the capacitor is -

$$v_c(t) = \frac{1}{C}\int i(t)dt$$

Differentiate the above equation with respect to time.

$$\frac{\mathrm{d}v_c(t)}{\mathrm{d}t} = \frac{i(t)}{C}$$

State vector, $X = \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$

Differential state vector, $\dot{X} = \begin{bmatrix} \frac{\mathrm{d}i(t)}{\mathrm{d}t} \\ \frac{\mathrm{d}v_c(t)}{\mathrm{d}t} \end{bmatrix}$

# Solution Example 1

We can arrange the differential equations and output equation into the standard form of state space model as,

$$\dot{X} = \begin{bmatrix} \frac{di(t)}{dt} \\ \frac{dv_c(t)}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix} [v_i(t)]$$

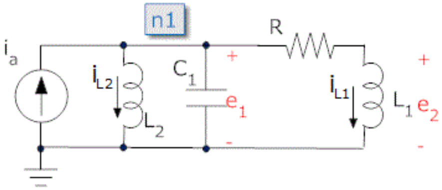
$$Y = [0 \quad 1] \begin{bmatrix} i(t) \\ v_c(t) \end{bmatrix}$$

Where,

$$A = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}, C = [0 \quad 1] \; and \; D = [0]$$

# Example 2

Derive a state space model for the system shown:



# Solution Example 2

There are three energy storage elements, so we expect three state equations. Try choosing $i_1$, $i_2$ and $e_1$ as state variables. Now we want equations for the derivatives. The voltage across the inductor $L_2$ is $e_1$ (which is one of our state variables)

$$L_2 \frac{di_{L2}}{dt} = e_1$$

so our first state variable equation is

$$\frac{di_{L2}}{dt} = \frac{1}{L_2} e_1$$

If we sum currents into the node labeled n1 we get

$$i_s - i_{L2} - i_{C1} - i_{L1} = 0$$

# Solution Example 2

This equation has our input (ia) and two state variable (iL2 and iL1) and the current through the capacitor. So from this we can get our second state equation

$$i_{C1} = C_1 \frac{de_1}{dt} = i_a - i_{L2} - i_{L1}$$

$$\frac{de_1}{dt} = \frac{1}{C_1}\left(i_a - i_{L2} - i_{L1}\right)$$

Our third, and final, state equation we get by writing an equation for the voltage across $L_1$ (which is $e_2$) in terms of our other state variables

$$e_2 = L_1 \frac{di_{L1}}{dt} = e_1 - Ri_{L1}$$

$$\frac{di_{L1}}{dt} = \frac{1}{L_1}\left(e_1 - Ri_{L1}\right)$$

We also need an output equation:

$$e_2 = e_1 - Ri_{L1}$$

# Solution Example 2

So our state space representation becomes

$$\mathbf{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix} = \begin{bmatrix} i_{L2} \\ e_1 \\ i_{L1} \end{bmatrix}$$

$$\dot{\mathbf{q}} = \mathbf{Aq} + \mathbf{B}u \qquad \mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L_2} & 0 \\ -\frac{1}{C_1} & 0 & -\frac{1}{C_1} \\ 0 & \frac{1}{L_1} & -\frac{R}{L_1} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{C_1} \\ 0 \end{bmatrix}$$

$$y = \mathbf{Cq} + Du \qquad \mathbf{C} = \begin{bmatrix} 0 & 1 & -R \end{bmatrix} \qquad D = 0$$

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# Example 3

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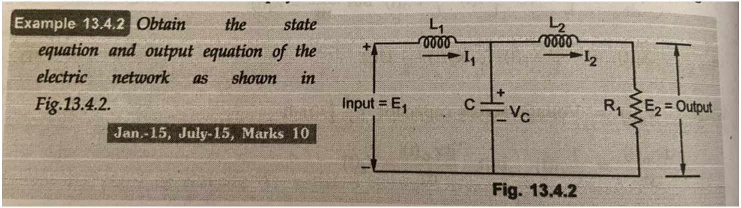

# Solution Example 3

**Solution :** The various currents are as shown in the Fig. 13.4.3.

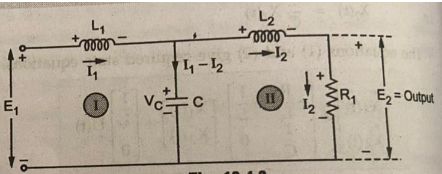


**Fig. 13.4.3**

Applying KVL to loop I and II,

$$-L_1\frac{dI_1}{dt} - V_C + E_1 = 0 \quad \text{i.e.}$$

$$\frac{dI_1}{dt} = -\frac{1}{L_1}V_C + \frac{1}{L_1}E_1 \quad ...(1)$$

$$-L_2\frac{dI_2}{dt} - I_2R_1 + V_C = 0 \quad \text{i.e.} \quad \frac{dI_2}{dt} = +\frac{1}{L_2}V_C - \frac{R_1}{L_2}I_2 \quad ...(2)$$

$$I_1 - I_2 = C\frac{dV_C}{dt} \quad \text{i.e.} \quad \frac{dV_C}{dt} = \frac{1}{C}I_1 - \frac{1}{C}I_2 \quad ...(3)$$

# Solution Example 3



And $E_2 = I_2 R_1$

Select state variables as, $I_1 = X_1$, $I_2 = X_2$, $V_C = X_3$ and input $E_1 = U$ and output $E_2 = Y$.

Using in the equations (1), (2), (3) and (4).

$$\dot{X}_1 = -\frac{1}{L_1}X_3 + \frac{1}{L_1}U$$

$$\dot{X}_2 = -\frac{R_1}{L_2}X_2 + \frac{1}{L_2}X_3, \quad \dot{X}_3 = \frac{1}{C}X_1 - \frac{1}{C}X_2$$

and $Y = X_2 R_1$

$\therefore$ $\dot{X} = AX + BU$ and $Y = CX + DU$

where
$$A = \begin{bmatrix} 0 & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_1}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}, \quad C = [0 \;\; R_1 \;\; 0], \; D = [0]$$

# STATE SPACE REPRESENTATION OF LINEAR DIFFERENTIAL EQUATION

Consider following nth order LTI system relating the output y(t) to the input u(t).

$$y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y^1 + a_n y = u$$

Phase variables: The phase variables are defined as those particular state variables which are obtained from one of the system variables & its (n-1) derivatives. Often the variables used is the system output & the remaining state variables are then derivatives of the output.

Let us define the state variables as

$$x_1 = y$$

$$x_2 = \frac{dy}{dt} = \frac{dx}{dt}$$

$$x_3 = \frac{d\dot{y}}{dt} = \frac{dx_2}{dt}$$

$$\vdots \quad \vdots \quad \vdots$$

$$x_n = y^{n-1} = \frac{dx_{n-1}}{dt}$$

# STATE SPACE REPRESENTATION OF LINEAR DIFFERENTIAL EQUATION

From the above equations we can write

$$\dot{x_1} = x_2$$

$$\dot{x_2} = x_3$$

$$\vdots \qquad \vdots$$

$$\dot{x_{n-1}} = x_n$$

$$\dot{x_n} = -a_n x_1 - a_{n-1} x_2 - \cdots \dots \dots \dots - a_1 x_n + u$$

Writing the above state equation in vector matrix form

$$\dot{X}(t) = AX(t) + Bu(t)$$

Where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ , $A = \begin{bmatrix} 0 & 1 & 0 \dots \dots & 0 \\ 0 & 0 & 1 \dots \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \dots \dots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} \dots \dots . & -a_1 \end{bmatrix}_{n\times n}$

# STATE SPACE REPRESENTATION OF LINEAR DIFFERENTIAL EQUATION



Writing the above state equation in vector matrix form

$$\dot{X}(t) = AX(t) + Bu(t)$$

$$\text{Where } X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}, \quad A = \begin{bmatrix} 0 & 1 & 0 \ldots \ldots & 0 \\ 0 & 0 & 1 \ldots \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 \ldots \ldots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} \ldots \ldots & -a_1 \end{bmatrix}_{n\times n}$$

Output equation can be written as

$$Y(t) = CX(t)$$

$$C = [1 \quad 0 \ldots \ldots \quad 0]_{1\times n}$$

# Example

transfer function.

**Example 13.6.1** *Construct the state model using phase variables if the system is described by the differential equation* : $\frac{d^3y(t)}{dt^3}+\frac{4d^2y(t)}{dt^2}+\frac{7dy(t)}{dt}+2y(t) = 5u(t)$. *Draw the state diagram.*

**Jan-15, 16, July-15, Marks 6**

# Example Solution

**Solution :** Select the state variables as,

$$y(t) = X_1, \quad \dot{X}_1 = X_2 = \frac{dy(t)}{dt}, \quad \dot{X}_2 = X_3 = \frac{d^2y(t)}{dt^2}, \quad \dot{X}_3 = \frac{d^3y(t)}{dt^3}$$

Using in the given equation, $\dot{X}_3 + 4X_3 + 7X_2 + 2X_1 = 5\,u(t)$

$$\therefore \quad \dot{X}_3 = -2X_1 - 7X_2 - 4X_3 + 5\,u(t) \quad \text{and} \quad y(t) = X_1$$

$$\dot{X}_1 = X_2, \quad \dot{X}_2 = X_3$$

Hence the state model using phase variables is,

$$\dot{X} = AX + BU \text{ and } Y = CX \text{ with,}$$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix}, \quad C = [1 \quad 0 \quad 0]$$

# State Space to Transfer Function

Consider the state space system:

$$\dot{\mathbf{q}}(t) = \mathbf{A}\mathbf{q}(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}\mathbf{q}(t) + Du(t)$$

Now, take the Laplace Transform (with zero initial conditions since we are finding a transfer function):

$$s\mathbf{Q}(s) = \mathbf{A}\mathbf{Q}(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}\mathbf{Q}(s) + DU(s)$$

We want to solve for the ratio of Y(s) to U(s), so we need so remove **Q**(s) from the output equation. We start by solving the state equation for **Q**(s)

$$s\mathbf{Q}(s) - \mathbf{A}\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\left(s\mathbf{I} - \mathbf{A}\right)\mathbf{Q}(s) = \mathbf{B}U(s)$$
$$\mathbf{Q}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B}U(s) = \mathbf{\Phi}(s)\mathbf{B}U(s); \quad \text{where } \mathbf{\Phi}(s) = \left(s\mathbf{I} - \mathbf{A}\right)^{-1}$$

# State Space to Transfer Function

$$\mathbf{Q}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s) = \mathbf{\Phi}(s)\mathbf{B}U(s); \quad \text{where } \mathbf{\Phi}(s) = (s\mathbf{I} - \mathbf{A})^{-1}$$

The matrix $\mathbf{\Phi}(s)$ is called the state transition matrix. Now we put this into the output equation

$$\begin{aligned} Y(s) &= \mathbf{C\Phi}(s)\mathbf{B}U(s) + DU(s) \\ &= (\mathbf{C\Phi}(s)\mathbf{B} + D)U(s) \end{aligned}$$

Now we can solve for the transfer function:

$$H(s) = \frac{Y(s)}{U(s)} = \mathbf{C\Phi}(s)\mathbf{B} + D = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + D$$

*Note that although there are many state space representations of a given system, all of those representations will result in the same transfer function (i.e., the transfer function of a system is unique; the state space representation is not).*

# Example

Find the transfer function of the system with state space representation

$$\dot{\mathbf{q}} = \mathbf{A}\mathbf{q} + \mathbf{B}\mathrm{u} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -3 & -4 & -2 \end{bmatrix}\mathbf{q} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\mathrm{u}$$

$$\mathrm{y} = \mathbf{C}\mathbf{q} + \mathrm{D}\mathrm{u} = \begin{bmatrix} 5 & 1 & 0 \end{bmatrix} + 0 \cdot \mathrm{u}$$

# Example Solution

First find (s**I**-**A**) and the $\mathbf{\Phi}$=(s**I**-**A**)$^{-1}$ (note: this calculation is not obvious. Details are here). Rules for inverting a 3x3 matrix are here.

$$s\mathbf{I}-\mathbf{A}=\begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 3 & 4 & s+2 \end{bmatrix}$$

$$\mathbf{\Phi}=(s\mathbf{I}-\mathbf{A})^{-1}=\frac{\begin{bmatrix} s^2+2s+4 & 2+s & 1 \\ -3 & s(2+s) & s \\ -3s & -3-4s & s^2 \end{bmatrix}}{s^3+2s^2+4s+3}$$

Now we can find the transfer function

$$H(s)=\frac{Y(s)}{U(s)}=\mathbf{C}\mathbf{\Phi}\mathbf{B}+D$$

$$=\frac{s+5}{s^3+2s^2+4s+3}$$


THANK YOU
YOU'RE AWESOME
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