Control Systems UNIT 4
Frequency Responses Analysis

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Introduction to Frequency Response Analysis

- We have already discussed time response analysis of the control systems and the time domain specifications of the second order control systems.
- In this unit, let us discuss the frequency response analysis of the control systems and the frequency domain specifications of the second order control systems.

What is Frequency Response?

- The response of a system can be partitioned into both the transient response and the steady state response. We can find the transient response by using Fourier integrals. The steady state response of a system for an input sinusoidal signal is known as the frequency response. In this chapter, we will focus only on the steady state response.
- If a sinusoidal signal is applied as an input to a Linear Time-Invariant (LTI) system, then it produces the steady state output, which is also a sinusoidal signal. The input and output sinusoidal signals have the same frequency, but different amplitudes and phase angles.

What is Frequency Response?

Let the input signal be –

$$r(t) = A\sin(\omega_0 t)$$

The open loop transfer function will be –

$$G(s) = G(j\omega)$$

We can represent $G(j\omega)$ in terms of magnitude and phase as shown below.

$$G(j\omega) = |G(j\omega)|\angle G(j\omega)$$

Substitute, $\omega = \omega_0$ in the above equation.

$$G(j\omega_0) = |G(j\omega_0)|\angle G(j\omega_0)$$

What is Frequency Response?

The output signal is

$$c(t) = A|G(j\omega_0)|\sin(\omega_0 t + \angle G(j\omega_0))$$

- The **amplitude** of the output sinusoidal signal is obtained by multiplying the amplitude of the input sinusoidal signal and the magnitude of $G(j\omega)$ at $\omega = \omega_0$.
- The **phase** of the output sinusoidal signal is obtained by adding the phase of the input sinusoidal signal and the phase of $G(j\omega)$ at $\omega = \omega_0$.

Where,

- **A** is the amplitude of the input sinusoidal signal.
- $\boldsymbol{\omega_0}$ is angular frequency of the input sinusoidal signal.

We can write, angular frequency ω_0 as shown below.

$$\omega_0 = 2\pi f_0$$

Magnitude and Phase $T(j\omega)$

Consider the transfer function of the second order closed loop control system as,

$$T(s) = \frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Substitute, $s = j\omega$ in the above equation.

$$T(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\delta\omega_n(j\omega) + \omega_n^2}$$

$$\Rightarrow T(j\omega) = \frac{\omega_n^2}{-\omega^2 + 2j\delta\omega\omega_n + \omega_n^2} = \frac{\omega_n^2}{\omega_n^2\left(1 - \frac{\omega^2}{\omega_n^2} + \frac{2j\delta\omega}{\omega_n}\right)}$$

$$\Rightarrow T(j\omega) = \frac{1}{\left(1 - \frac{\omega^2}{\omega_n^2}\right) + j\left(\frac{2\delta\omega}{\omega_n}\right)}$$

Magnitude and Phase $T(j\omega)$

Let, $\frac{\omega}{\omega_n} = u$ Substitute this value in the above equation.

$$T(j\omega) = \frac{1}{(1-u^2) + j(2\delta u)}$$

Magnitude of $T(j\omega)$ is -

$$M = |T(j\omega)| = \frac{1}{\sqrt{(1-u^2)^2 + (2\delta u)^2}}$$

Phase of $T(j\omega)$ is -

$$\angle T(j\omega) = -tan^{-1}\left(\frac{2\delta u}{1-u^2}\right)$$

Resonant Frequency

It is the frequency at which the magnitude of the frequency response has peak value for the first time. It is denoted by ω_r . At $\omega = \omega_r$, the first derivate of the magnitude of $T(j\omega)$ is zero.

$$\Rightarrow \omega_r = \omega_n \sqrt{1 - 2\delta^2}$$

Resonant Peak

It is the peak (maximum) value of the magnitude of $T(j\omega)$. It is denoted by M_r.

$$\Rightarrow M_r = \frac{1}{2\delta\sqrt{1-\delta^2}}$$

Resonant peak in frequency response corresponds to the peak overshoot in the time domain transient response for certain values of damping ratio δ. So, the resonant peak and peak overshoot are correlated to each other.

Bandwidth

It is the range of frequencies over which, the magnitude of $T(j\omega)$ drops to 70.7% from its zero frequency value.

$$\Rightarrow \omega_b = \omega_n \sqrt{1 - 2\delta^2 + \sqrt{(2 - 4\delta^2 + 4\delta^4)}}$$

Bandwidth ω_b in the frequency response is inversely proportional to the rise time t_r in the time domain transient response.

Bode Plots

The Bode plot or the Bode diagram consists of two plots –

- Magnitude plot
- Phase plot

In both the plots, x-axis represents angular frequency (logarithmic scale). Whereas, yaxis represents the magnitude (linear scale) of open loop transfer function in the magnitude plot and the phase angle (linear scale) of the open loop transfer function in the phase plot.

The **magnitude** of the open loop transfer function in dB is -

$$M = 20\ \log |G(j\omega)H(j\omega)|$$

The **phase angle** of the open loop transfer function in degrees is -

$$\phi = \angle G(j\omega)H(j\omega)$$

General procedure for constructing the Bode plots(Basic factors)

Type of term	G(jω)H(jω)	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
Constant	K	0	$20\log K$	0
Zero at origin	$j\omega$	20	$20\log\omega$	90
'n' zeros at origin	$(j\omega)^n$	$20\,n$	$20\,n\log\omega$	$90\,n$
Pole at origin	$\frac{1}{j\omega}$	-20	$-20\log\omega$	$-90\ or\ 270$

General procedure for constructing the Bode plots(Basic factors)

Type of term	G(jω)H(jω)	Slope(dB/dec)	Magnitude (dB)	Phase angle(degrees)
'n' poles at origin	$\frac{1}{(j\omega)^n}$	$-20\,n$	$-20\,n\log\omega$	$-90\,n$ *or* $270\,n$
Simple zero	$1+j\omega r$	20	0 *for* $\omega<\frac{1}{r}$ $20\,\log\omega r$ *for* $\omega>\frac{1}{r}$	0 *for* $\omega<\frac{1}{r}$ 90 *for* $\omega>\frac{1}{r}$
Simple pole	$\frac{1}{1+j\omega r}$	-20	0 *for* $\omega<\frac{1}{r}$ $-20\,\log\omega r$ *for* $\omega>\frac{1}{r}$	0 *for* $\omega<\frac{1}{r}$ -90 *or* 270 *for* $\omega>\frac{1}{r}$

General procedure for constructing the Bode plots(Basic factors)

Consider the open loop transfer function $G(s)H(s) = K$.

Magnitude $M = 20 \log K$ dB

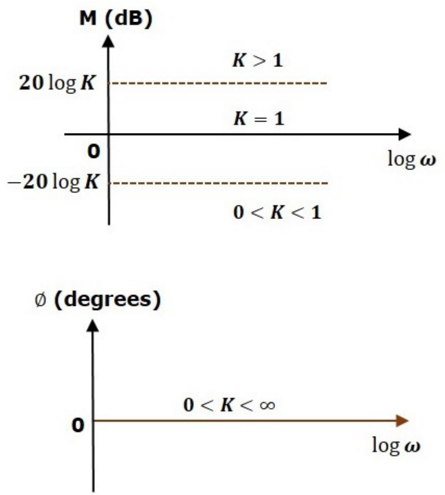
Phase angle $\phi = 0$ degrees

If $K = 1$, then magnitude is 0 dB.

If $K > 1$, then magnitude will be positive.

If $K < 1$, then magnitude will be negative.

General procedure for constructing the Bode plots(Basic factors)



General procedure for constructing the Bode plots(Basic factors)

The magnitude plot is a horizontal line, which is independent of frequency. The 0 dB line itself is the magnitude plot when the value of K is one. For the positive values of K, the horizontal line will shift $20 \log K$ dB above the 0 dB line. For the negative values of K, the horizontal line will shift $20 \log K$ dB below the 0 dB line. The Zero degrees line itself is the phase plot for all the positive values of K.

Consider the open loop transfer function $G(s)H(s) = s$.

Magnitude $M = 20 \log \omega$ dB

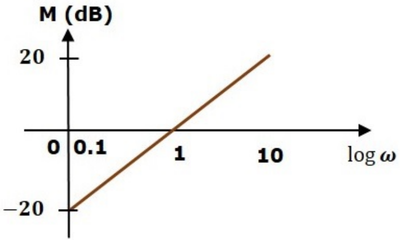
Phase angle $\phi = 90^0$

General procedure for constructing the Bode plots(Basic factors)

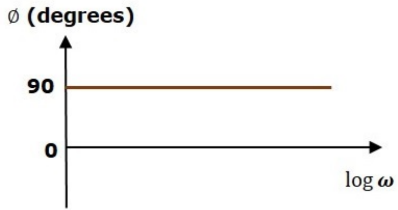
At $\omega = 1$ rad/sec, the magnitude is 0 dB.

At $\omega = 10$ rad/sec, the magnitude is 20 dB.

The following figure shows the corresponding Bode plot.



General procedure for constructing the Bode plots(Basic factors)



The magnitude plot is a line, which is having a slope of 20 dB/dec. This line started at $\omega = 0.1$ rad/sec having a magnitude of -20 dB and it continues on the same slope. It is touching 0 dB line at $\omega = 1$ rad/sec. In this case, the phase plot is 90^0 line.

Rules for Construction of Bode Plots

1) Determine the Transfer Function of the system:

$$H(s) = \frac{K(s+z_1)}{s(s+p_1)}$$

2) Rewrite it by factoring both the numerator and denominator into the **standard** form

$$H(s) = \frac{Kz_1(s/z_1 + 1)}{sp_1(s/p_1 + 1)}$$

where the z s are called zeros and the p s are called poles.

Rules for Construction of Bode Plots

3) Replace *s* with ***j?*** . Then find the **Magnitude** of the Transfer Function.

$$H(jw) = \frac{Kz_1(\frac{jw}{z_1}+1)}{jwp_1(\frac{jw}{p_1}+1)}$$

If we take the $\log_{10}$ of this magnitude and multiply it by 20 it takes on the form of

$$20\log_{10}(H(jw)) = 20\log_{10}\left(\frac{Kz_1(\frac{jw}{z_1}+1)}{jwp_1(\frac{jw}{p_1}+1)}\right) =$$

$$20\log_{10}|K| + 20\log_{10}|z_1| + 20\log_{10}\left|(\frac{jw}{z_1}+1)\right| - 20\log_{10}|p_1| - 20\log_{10}|jw| - 20\log_{10}\left|(\frac{jw}{z_1}+1)\right|$$

Each of these individual terms is very easy to show on a logarithmic plot. The entire Bode log magnitude pl the result of the superposition of all the straight line terms. This means with a little practice, we can quickly the effect of each term and quickly find the overall effect. To do this we have to understand the effect of the different types of terms.

Rules for Construction of Bode Plots

Each of these individual terms is very easy to show on a logarithmic plot. The entire Bode log magnitude pl the result of the superposition of all the straight line terms. This means with a little practice, we can quickly the effect of each term and quickly find the overall effect. To do this we have to understand the effect of the different types of terms.

These include: 1) Constant terms K

2) Poles and Zeros at the origin $|j?|$

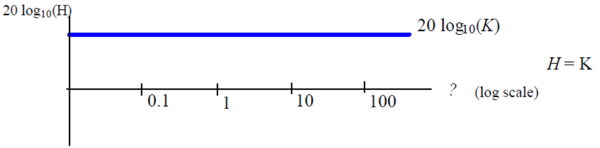
3) Poles and Zeros not at the origin $\left|1+\frac{j\omega}{p_1}\right|$ or $\left|1+\frac{j\omega}{z_1}\right|$

4) Complex Poles and Zeros (addressed later)

Effect of Constant Terms

Effect of Constant Terms:

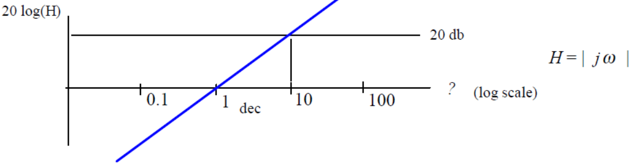
Constant terms such as K contribute a straight horizontal line of magnitude $20\log_{10}(K)$



Effect of Individual Zeros and Poles at the Origin

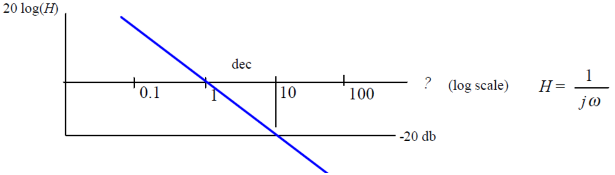
Effect of Individual Zeros and Poles at the origin:

A **zero** at the origin occurs when there is an *s* or *j?* multiplying the numerator. Each occurrence of this causes a positively sloped line passing through ? = 1 with a rise of 20 db over a decade.



A Pole at the Origin

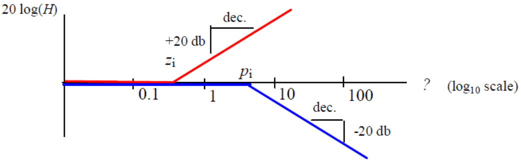
A pole at the origin occurs when there are *s* or ***j?*** multiplying the denominator. Each occurrence of this causes a negatively sloped line passing through ? = 1 with a drop of 20 db over a decade.



Effect of Individual Zeros and Poles Not at the Origin

Effect of Individual Zeros and Poles Not at the Origin

Zeros and Poles **not at the origin** are indicated by the **(1+*j*? /z_i)** and **(1+*j*? /p_i)**. The values z_i and p_i in each of these expression is called a **critical frequency** (or break frequency). Below their critical frequency these terms do not contribute to the log magnitude of the overall plot. Above the critical frequency, they represent a ramp function of 20 db per decade. Zeros give a positive slope. Poles produce a negative slope.



$$H = \frac{\left|1 + \frac{j\omega}{z_i}\right|}{\left|1 + \frac{j\omega}{p_i}\right|}$$

Example 1

For the transfer function given, sketch the Bode log magnitude diagram which shows how the log magnitude of the system is affected by changing input frequency. (*TF=transfer function*)

$$TF = \frac{1}{2s + 100}$$

Example 1 Solution

Step 1: Repose the equation in Bode plot form:

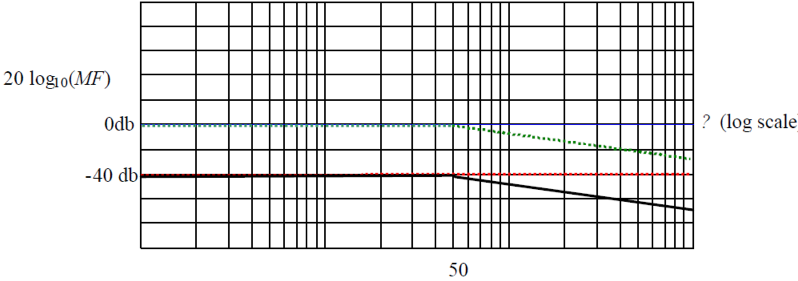
$$TF = \frac{\left(\frac{1}{100}\right)}{\frac{s}{50}+1} \qquad \text{recognized as} \qquad TF = \frac{K}{\frac{1}{p_1}s+1}$$

with $K = 0.01$ and $p_1 = 50$

For the constant, K: $20\log_{10}(0.01) = -40$

For the pole, with critical frequency, p_1:

Example 1 Solution



Example 2

Your turn. Find the Bode log magnitude plot for the transfer function,

$$TF = \frac{5x10^4 s}{s^2 + 505s + 2500}$$

Start by simplifying the transfer function form:

Example 2 Solution

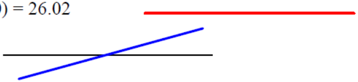
$$TF = \frac{5x10^4 s}{s^2 + 505s + 2500}$$

Simplify transfer function form:

$$TF = \frac{5x10^4 \quad s}{(s+5)(s+500)} = \frac{\frac{5x10^4}{5*500} \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)} = \frac{20 \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)}$$

Example 2 Solution

Recognize: $K = 20$ → $20 \log 10(20) = 26.02$



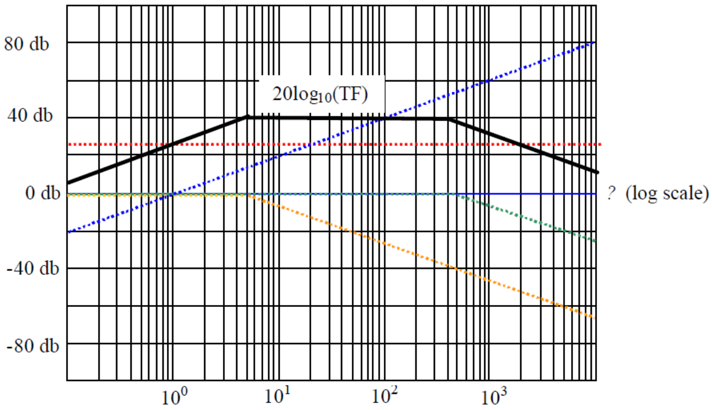
1 zero at the origin

2 poles: at $p_1 = 5$ and p_2=500

Technique to get started:

1) Draw the line of each individual term on the graph
2) Follow the combined **pole-zero at the origin** line back to the left side of the graph.
3) Add the constant offset, **20 log$_{10}$(K),** to the value where the **pole/zero at the origin** line intersects the le side of the graph.
4) Apply the effect of the **poles/zeros not at the origin**. working from left (low values) to right (higher values) of the poles/zeros.

Example 2 Solution



Example 3

Example 3: One more time. This one is harder. Find the Bode log magnitude plot for the transfer function,

$$TF = \frac{200(s+20)}{s(2s+1)(s+40)}$$

Example 3 Solution

$$TF = \frac{200(s+20)}{s(2s+1)(s+40)}$$

Simplify transfer function form:

$$TF = \frac{200(s+20)}{s(2s+1)(s+40)} = \frac{\frac{200*20}{40}\left(\frac{s}{20}+1\right)}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{40}+1\right)} = \frac{100\left(\frac{s}{20}+1\right)}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{40}+1\right)}$$

Example 3 Solution

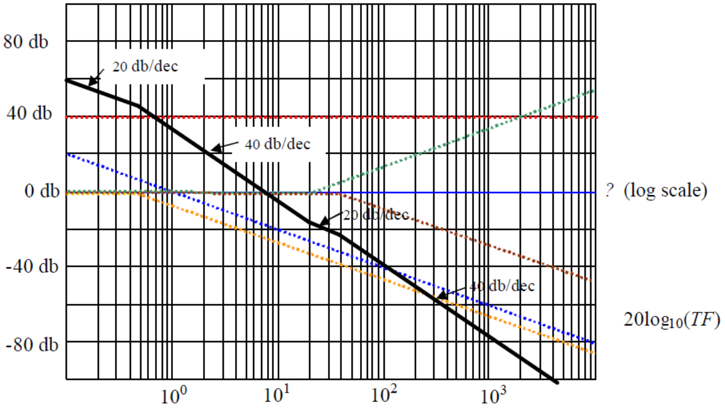
Recognize: $K = 100$ → $20\log10(100) = 40$

1 pole at the origin

1 zero at $z_1 = 20$

2 poles: at $p_1 = 0.5$ and p_2=40

Example 3 Solution



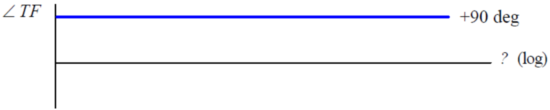
Phase Bode Plot

Effect of Constants on Phase:

A **positive** constant, ***K*>0,** has no effect on phase. A **negative** constant, ***K*<0,** will set up a phase shift of $\pm 180^{\circ}$. (Remember real vs imaginary plots – a negative real number is at $\pm 180^{\circ}$ relative to the origin)

Effect of Zeros at the origin on Phase Angle:

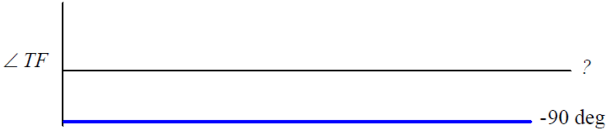
Zeros at the origin, ***s*,** cause a constant **+90 degree shift** for each zero.



Phase Bode Plot

Effect of Poles at the origin on Phase Angle:

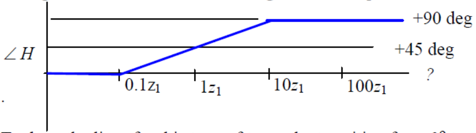
Poles at the origin, s^{-1}, cause a constant **-90 degree shift** for each pole.



Phase Bode Plot

Effect of Zeros not at the origin on Phase Angle:

Zeros not at the origin, like $\left|1 + \frac{j\omega}{z_1}\right|$, have no phase shift for frequencies **much lower** than z_i, have a +45 deg shift at z_1, and have a +90 deg shift for frequencies **much higher** than z_1.

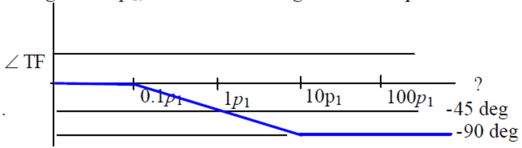


To draw the lines for this type of term, the transition from 0° to +90° is drawn over 2 decades, starting at $0.1z_1$ and ending at $10z_1$.

Phase Bode Plot

Poles not at the origin, like $\frac{1}{\left|1+\frac{j\omega}{p_1}\right|}$, have no phase shift for frequencies **much lower** than p

45 deg shift at p_1, and have a -90 deg shift for frequencies **much higher** than p_1.



To draw the lines for this type of term, the transition from 0° to -90° is drawn over 2 decades, start
$0.1p_1$ and ending at $10p_1$.

Phase Bode Plot

Poles not at the origin, like $\frac{1}{\left|1+\frac{j\omega}{p_1}\right|}$, have no phase shift for frequencies **much lower** than p

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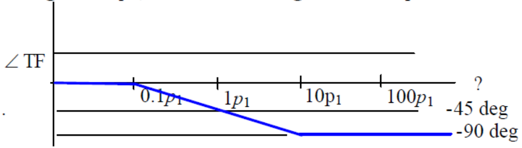


To draw the lines for this type of term, the transition from 0° to -90° is drawn over 2 decades, start
$0.1p_1$ and ending at $10p_1$.

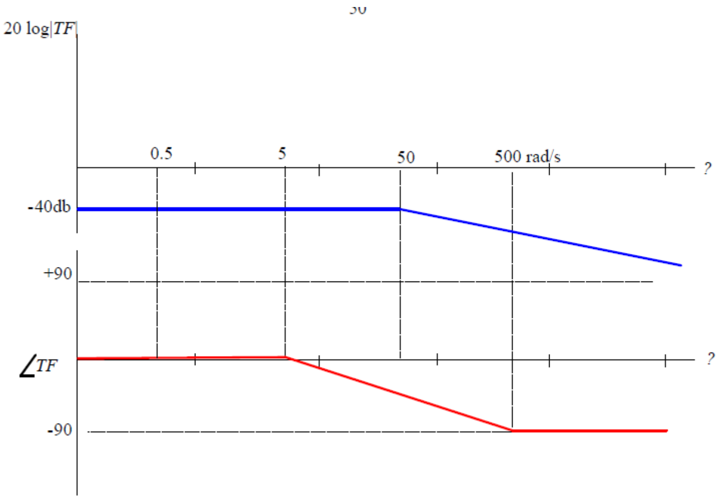
Example 1

Example 1:

For the Transfer Function given, sketch the Bode diagram which shows how the phase of the system is affected by changing input frequency.

$$TF = \frac{1}{2s+100} = \frac{(1/100)}{(\frac{s}{50}+1)}$$

Solution Example 1



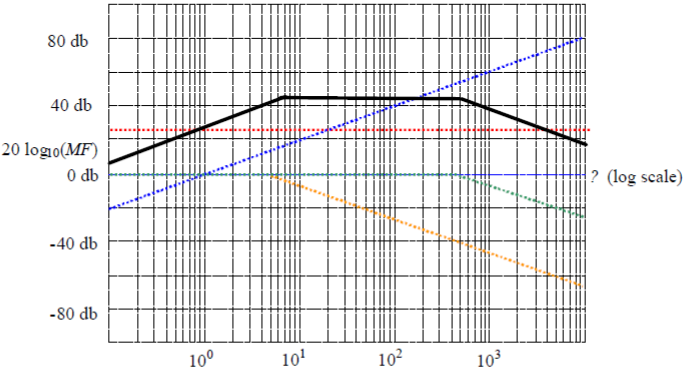
Example 2

Repeat for the transfer function,

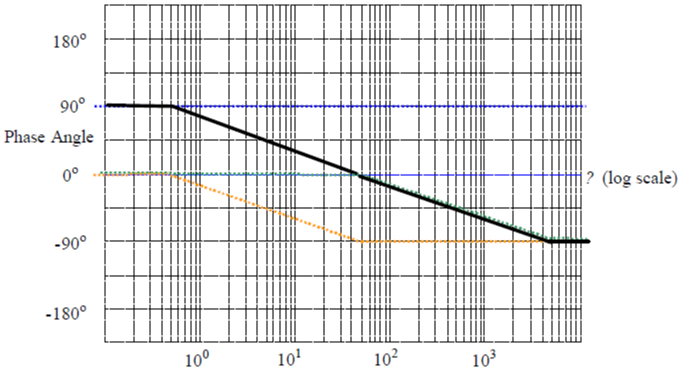
$20\log|TF|$

$$TF = \frac{5x10^4 s}{s^2 + 505s + 2500} = \frac{20 \quad s}{(\frac{s}{5}+1)(\frac{s}{500}+1)}$$

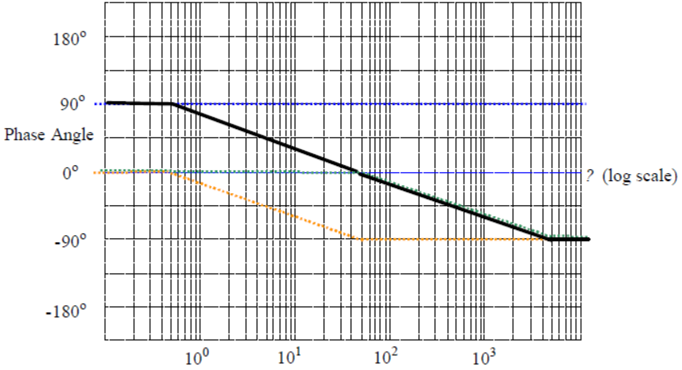
Solution Example 2



Solution Example 2



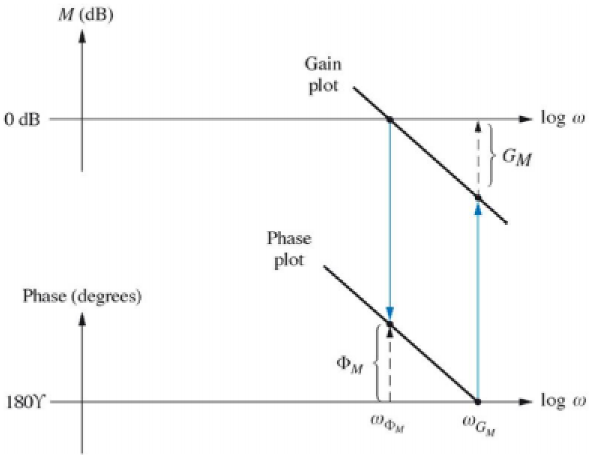
Solution Example 2



Gain Margin

- The greater the Gain Margin (GM), the greater the stability of the system. The gain margin refers to the amount of gain, which can be increased or decreased without making the system unstable. It is usually expressed as a magnitude in dB.
- The frequency where the Bode phase plot = 180. This point is known as the phase crossover frequency.
- The Gain margin is the difference between the magnitude curve and 0dB at the point corresponding to the frequency that gives us a phase of -180 deg (the phase cross over frequency, Wpc).

Gain Margin


M (dB)
Gain plot
0 dB
log ω
G_M
Phase plot
Phase (degrees)
Φ_M
180Y
log ω
ω_{Φ_M}
ω_{G_M}

Gain Margin

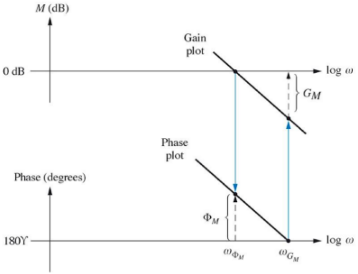
The **Phase Crossover Frequency**, ω_{pc} is the frequency (frequencies) at which $\angle G(\imath\omega_{pc}) = 180°$.

Definition 5.

The **Gain Margin**, G_M is the gain relative to $0dB$ when $\angle G = 180°$.

- $G_M = -20\log|G(\imath\omega_{pc})|$

G_M is the gain (in dB) which will destabilize the system in closed loop.



Phase Margin

- The phase margin is defined as the change in open loop phase shift required to make a closed loop system unstable
- The phase margin is the difference in phase between the phase curve and -180 deg at the point corresponding to the frequency that gives us a gain of 0dB (the gain cross over frequency, Wgc).

Phase Margin

The **Gain Crossover Frequency**, ω_{gc} is the frequency at which $|G(\imath\omega_c)| = 1$.

The **Phase Margin**, Φ_M is the phase relative to 180° when $|G| = 1$.

- $\Phi_M = |180^\circ - \angle G(i\omega_{gc})|$

Bode Plot Stability

- For a Stable System: Both the margins should be positive or phase margin should be greater than the gain margin.
- For Marginal Stable System: Both the margins should be zero or phase margin should be equal to the gain margin.
- For Unstable System: If any of them is negative or phase margin should be less than the gain margin.

Example

Example 11.6.3 *Find the gain margin and phase margin analytically for the negative feedback control system having open loop TF,* $G(s)H(s) = \dfrac{6}{(s^2+2s+2)(s+2)}$.

Dec.-11, Jan.-06, Marks 8

Example Solution

Example 11.6.3 *Find the gain margin and phase margin analytically for the negative feedback control system having open loop TF,* $G(s)H(s) = \dfrac{6}{(s^2+2s+2)(s+2)}$.

Dec.-11, Jan.-06, Marks 8

Solution : $G(j\omega)H(j\omega) = \dfrac{6}{\left[(j\omega)^2+2j\omega+2\right]\left[j\omega+2\right]} = \dfrac{6}{\left[(2-\omega^2)+j2\omega\right]\left[2+j\omega\right]}$

Example Solution

$$M_R = \frac{6}{\sqrt{(2-\omega^2)^2 + 4\omega^2}\,\sqrt{4+\omega^2}}$$

For ω_{gc}, $M_R = 1$ i.e. $\dfrac{6}{\sqrt{(2-\omega^2)^2 + 4\omega^2}\,\sqrt{4+\omega^2}} = 1$

$\therefore \quad 36 = \left[(2-\omega^2)^2 + 4\omega^2\right]\left[4+\omega^2\right]$ i.e. $\omega^6 + 4\omega^2 + 4\omega^4 = 20$

Solving, $\omega_{gc} = \mathbf{1.2528\ rad/sec}$

$$P.M. = 180^\circ + \angle G(j\omega)H(j\omega)\big|_{\omega=\omega_{gc}}$$

$$= 180^\circ - \tan^{-1}\left\{\frac{2\omega}{(2-\omega^2)}\right\} - \tan^{-1}\left(\frac{\omega}{2}\right)\bigg|_{\omega=1.2528} = +67.68^\circ$$

Example Solution

For ω_{pc}, rationalize $G(j\omega)H(j\omega)$ and equate imaginary part to zero.

$$G(j\omega)H(j\omega) = \frac{6\left[(2-\omega^2)-j2\omega\right][2-j\omega]}{\left[(2-\omega^2)+j2\omega\right]\left[(2-\omega^2)-j2\omega\right][2+j\omega][2-j\omega]}$$

$$= \frac{6\left[4-4\omega^2\right]+6j\omega(\omega^2-6)}{\left[(2-\omega^2)^2+4\omega^2\right]\left[4+\omega^2\right]} \quad \text{i.e} \quad \omega(\omega^2-6)=0$$

For $\omega = 0$, $\angle G(j\omega)H(j\omega) = 0^\circ$ hence ω_{pc} cannot be zero.

$$\therefore \quad \omega^2 = 6 \quad \text{i.e.} \quad \omega_{pc} = \sqrt{6} \ \text{rad/sec}$$

$$\text{G.M.} = 20\log\left[\frac{1}{\left|G(j\omega)H(j\omega)\right|_{\omega=\omega_{pc}}}\right]$$

$$\left|G(j\omega)H(j\omega)\right|_{\omega=\omega_{pc}} = \frac{6}{\sqrt{(16+24)}\times\sqrt{4+6}} = 0.3$$

$$\text{G.M.} = 20\log\frac{1}{0.3} = +\,10.457\ \text{dB}$$

Example

Example 11.6.1 *The Fig. 11.6.2 shows a block diagram of a vehicle control system. Determine the gain K such that the phase margin is 50°. What is the gain margin in this case ?*

July-04, Marks 8

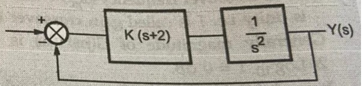


Fig. 11.6.2

Example Solution

Solution : From the given system,

$$G(s)H(s) = \frac{K(s+2)}{s^2} \quad \text{i.e.} \quad G(j\omega)H(j\omega) = \frac{K(2+j\omega)}{(j\omega)^2}$$

$$P.M. = 180^\circ + \angle G(j\omega)H(j\omega)\big|_{\omega=\omega_{gc}}$$

$$\therefore \quad 50^\circ = 180^\circ + \left[+\tan^{-1}\frac{\omega}{2} - 180^\circ\right]\Big|_{\omega=\omega_{gc}}$$

$$\therefore \quad \tan^{-1}\frac{\omega_{gc}}{2} = 50^\circ \quad \text{i.e.} \quad \frac{\omega_{gc}}{2} = \tan 50^\circ = 1.1917$$

$$\therefore \quad \omega_{gc} = 2 \times 1.1917 = 2.3835 \text{ rad/sec}$$

At ω_{gc},
$$|G(j\omega)H(j\omega)| = 1 \quad \text{i.e.} \quad \frac{|K|\sqrt{4+\omega_{gc}^2}}{(\omega_{gc})^2} = 1$$

$$K = 1.8258 \quad \text{for} \quad P.M. = +50^\circ$$

Example Solution

$$\therefore \quad \omega_{gc} = 2 \times 1.1917 = 2.3835 \text{ rad/sec}$$

At ω_{gc}, $|G(j\omega)H(j\omega)| = 1$ i.e. $\dfrac{|K|\sqrt{4+\omega_{gc}^2}}{(\omega_{gc})^2} = 1$

$$\therefore \quad K = 1.8258 \quad \text{for} \quad \text{P.M.} = +50^\circ$$

To find ω_{pc}, rationalise $G(j\omega)H(j\omega)$.

$$G(j\omega)H(j\omega) = \frac{K(2+j\omega)}{-\omega^2} = \frac{-2K}{\omega^2} - j\frac{K}{\omega}$$

Thus $\omega_{pc} \rightarrow \infty$ to make imaginary part zero.

$$\text{G.M.} = +\infty \text{ dB for the system.}$$

Example

xample 11.6.2 *For the loop gain function given below, determine the value of K analytically, so that the gain margin is 12 dB.*

$$G(s)H(s) = \frac{K}{s(s^2+13s+121)}$$

Further, for this value of K, find analytically (without plotting Bode plot), the ph

Aug.-06, Marks

margin.

Example Solution

Solution : $G(j\omega)H(j\omega) = \dfrac{K}{j\omega[(j\omega)^2 + 13j\omega + 121]} = \dfrac{K}{j\omega\,[(121-\omega^2) + j13\omega]}$

$$\text{G.M.} = 20\log\left\{\frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}\right\}$$

To find ω_{pc}, **rationalize** $G(j\omega)H(j\omega)$,

$$G(j\omega)H(j\omega) = \frac{K[-j\omega][(121-\omega^2)-j13\omega]}{j\omega(-j\omega)[(121-\omega^2)+j13\omega][(121-\omega^2)-j13\omega]}$$

Example Solution

$$= \frac{-j\omega K[121-\omega^2 - j13\omega]}{\omega^3[(121-\omega^2)^2+169\omega^2]} = \frac{-13K}{[(121-\omega^2)^2+169\omega^2]} - \frac{j\omega K(121-\omega^2)}{[\omega^2][(121-\omega^2)^2+169\omega^2]}$$

Equating imaginary part to zero,

$$121-\omega^2 = 0 \text{ i.e. } \omega_{pc} = \sqrt{121} = 11 \text{ rad / sec}$$

$$\therefore |G(j\omega)H(j\omega)|_{\omega=\omega_{pc}} = \left| \frac{-13K}{\{[121-(11)^2]^2 + 169\times 121\}} \right| = \frac{K}{1573}$$

Now given G.M. = + 12 dB,

$$\therefore \quad 12 = 20\log\left\{\frac{1}{\left(\frac{K}{1573}\right)}\right\} \quad \text{i.e} \quad \frac{12}{20} = \log\left\{\frac{1573}{K}\right\}$$

$$\therefore \quad 3.9810 = \frac{1573}{K} \quad \text{i.e.} \quad K = 395.119$$ ....for G.M.

Now find ω_{gc} to find phase margin with K = 395.119

Example Solution

Now find ω_{gc} to find phase margin with K = 395.119

At $\omega = \omega_{gc}$, $|G(j\omega)H(j\omega)| = 1$

$$\therefore \quad \frac{395.119}{|j\omega|\,|(121-\omega^2)+j13\omega|} = 1 \quad \text{i.e.} \quad \frac{395.119}{\omega \times \sqrt{(121-\omega^2)^2+169\omega^2}} = 1$$

$$\therefore \quad 156119.0242 = \omega^2\,[(121-\omega^2)^2+169\omega^2]$$

By trial and error, ω_{gc} = 3.35 rad/sec

$$P.M. = 180^\circ + \angle G(j\omega)H(j\omega)\big|_{\omega=\omega_{gc}} = 180^\circ + \left\{\frac{\angle K}{\angle j\omega \angle \{[121-(3.35)^2]+j13\times 3.35\}}\right\}$$

$$= 180^\circ + \left\{\frac{0^\circ}{90^\circ\ 21.638^\circ}\right\} = 180^\circ - 111.638^\circ = +\ 68.36^\circ$$


THANK YOU
YOU'RE AWESOME
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