Control Systems UNIT 3 Root Locus

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Introduction

- Root locus analysis is a graphical method for examining how the roots of a system change with variation of a certain system parameter, commonly the gain of a feedback system.
- This is a technique used in the field of control systems developed by Walter R. Evans.
- The Root Locus technique can be used to give graphical representation of a systems stability. We can see clearly the ranges of stability, ranges of instability, and the conditions that cause a system to break into oscillation.
- The root locus is a graphical representation in s-domain and it is symmetrical about the real axis.

Root locus concepts

The Root locus is the locus of the roots of the characteristic equation by varying system gain K from zero to infinity.

We know that, the characteristic equation of the closed loop control system is

$$1 + G(s)H(s) = 0$$

We can represent $G(s)H(s)$ as

$$G(s)H(s) = K\frac{N(s)}{D(s)}$$

Where,

- K represents the multiplying factor
- N(s) represents the numerator term having (factored) n^{th} order polynomial of 's'.
- D(s) represents the denominator term having (factored) m^{th} order polynomial of 's'.

Root locus concepts

Substitute, $G(s)H(s)$ value in the characteristic equation.

$$1 + k\frac{N(s)}{D(s)} = 0$$

$$\Rightarrow D(s) + KN(s) = 0$$

Case 1 – K = 0

If $K = 0$, then $D(s) = 0$.

That means, the closed loop poles are equal to open loop poles when K is zero.

Case 2 – K = ∞

Re-write the above characteristic equation as

$$K\left(\frac{1}{K} + \frac{N(s)}{D(s)}\right) = 0 \Rightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$$

Root locus concepts

Case 2 – K = ∞

Re-write the above characteristic equation as

$$K\left(\frac{1}{K}+\frac{N(s)}{D(s)}\right)=0 \Rightarrow \frac{1}{K}+\frac{N(s)}{D(s)}=0$$

Substitute, $K=\infty$ in the above equation.

$$\frac{1}{\infty}+\frac{N(s)}{D(s)}=0 \Rightarrow \frac{N(s)}{D(s)}=0 \Rightarrow N(s)=0$$

If $K=\infty$, then $N(s)=0$. It means the closed loop poles are equal to the open loop

zeros when K is infinity.

From above two cases, we can conclude that the root locus branches start at open loop poles and end at open loop zeros.

Angle Condition and Magnitude Condition

The points on the root locus branches satisfy the angle condition. So, the angle condition is used to know whether the point exist on root locus branch or not. We can find the value of K for the points on the root locus branches by using magnitude condition. So, we can use the magnitude condition for the points, and this satisfies the angle condition.

Characteristic equation of closed loop control system is

$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1 + j0$$

The **phase angle** of $G(s)H(s)$ is

$$\angle G(s)H(s) = \tan^{-1}\left(\frac{0}{-1}\right) = (2n+1)\pi$$

Angle Condition and Magnitude Condition

The **angle condition** is the point at which the angle of the open loop transfer function is an odd multiple of 180^0.

Magnitude of $G(s)H(s)$ is -

$$|G(s)H(s)| = \sqrt{(-1)^2 + 0^2} = 1$$

The magnitude condition is that the point (which satisfied the angle condition) at which the magnitude of the open loop transfer function is one.

Rules for Construction of Root Locus

Rule 1 – Locate the open loop poles and zeros in the 's' plane.

Rule 2 – Find the number of root locus branches.

We know that the root locus branches start at the open loop poles and end at open loop zeros. So, the number of root locus branches **N** is equal to the number of finite open loop poles **P** or the number of finite open loop zeros **Z**, whichever is greater.

Mathematically, we can write the number of root locus branches **N** as

$$N = P \quad \text{if} \quad P \geq Z$$

$$N = Z \quad \text{if} \quad P < Z$$

Rule 3: A point on the real axis lies on the root locus if the sum of number of open loop poles and the open loop zeros, on the real axis to the right hand side of this point is odd.

Rules for Construction of Root Locus

Example

Let us now draw the root locus of the control system having open loop transfer function,

$$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$$

Rules for Construction of Root Locus

Example

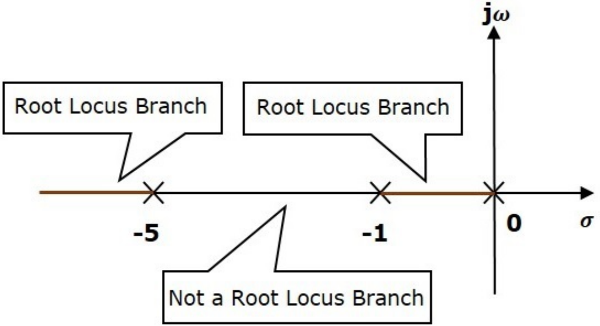
Let us now draw the root locus of the control system having open loop transfer function,

$G(s)H(s) = \frac{K}{s(s+1)(s+5)}$

Step 1 – The given open loop transfer function has three poles at $s = 0, s = -1$ and $s = -5$. It doesn't have any zero. Therefore, the number of root locus branches is equal to the number of poles of the open loop transfer function.

$$N = P = 3$$

Rules for Construction of Root Locus



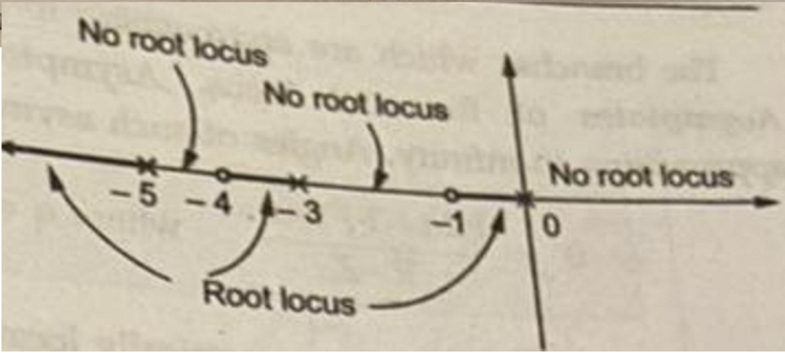
The three poles are located are shown in the above figure. The line segment between $s = -1$ and $s = 0$ is one branch of root locus on real axis. And the other branch of the root locus on the real axis is the line segment to the left of $s = -5$.

Rules for Construction of Root Locus

To understand this rule consider the following example 9.5.1.

Example 9.5.1 $G(s)H(s) = \frac{K(s+1)(s+4)}{s(s+3)(s+5)}$. *Using Rule No. 3, find on which se... axis the root locus exists.*

Sol



Rules for Construction of Root Locus

Rule 4 – Find the centroid and the angle of asymptotes.

- If $P = Z$, then all the root locus branches start at finite open loop poles and end at finite open loop zeros.
- If $P > Z$, then Z number of root locus branches start at finite open loop poles and end at finite open loop zeros and $P - Z$ number of root locus branches start at finite open loop poles and end at infinite open loop zeros.
- If $P < Z$, then P number of root locus branches start at finite open loop poles and end at finite open loop zeros and $Z - P$ number of root locus branches start at infinite open loop poles and end at finite open loop zeros.

Rules for Construction of Root Locus

So, some of the root locus branches approach infinity, when $P \neq Z$. Asymptotes give the direction of these root locus branches. The intersection point of asymptotes on the real axis is known as **centroid**.

We can calculate the **centroid α** by using this formula,

$$\alpha = \frac{\sum Real\ part\ of\ finite\ open\ loop\ poles - \sum Real\ part\ of\ finite\ open\ loop\ zeros}{P - Z}$$

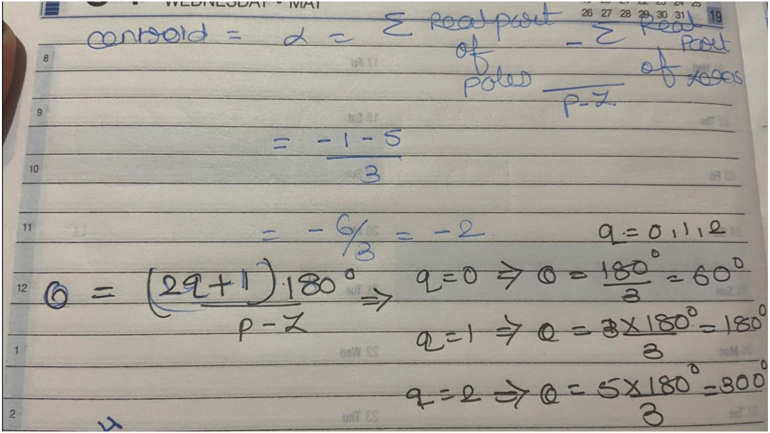
The formula for the angle of **asymptotes θ** is

$$\theta = \frac{(2q+1)180^0}{P - Z}$$

Where,

$$q = 0, 1, 2, \ldots, (P - Z) - 1$$

Rules for Construction of Root Locus



centroid = $\alpha = \dfrac{\sum \text{Real part of poles} - \sum \text{Real part of zeros}}{P-Z}$

$= \dfrac{-1-5}{3}$

$= -6/3 = -2$

$\theta = \dfrac{(2q+1)\cdot 180^\circ}{P-Z} \Rightarrow$ $q = 0,1,2$

$q=0 \Rightarrow \theta = \dfrac{180^\circ}{3} = 60^\circ$

$q=1 \Rightarrow \theta = \dfrac{3\times 180^\circ}{3} = 180^\circ$

$q=2 \Rightarrow \theta = \dfrac{5\times 180^\circ}{3} = 300^\circ$

Rules for Construction of Root Locus

Step 2 – We will get the values of the centroid and the angle of asymptotes by using the given formulae.

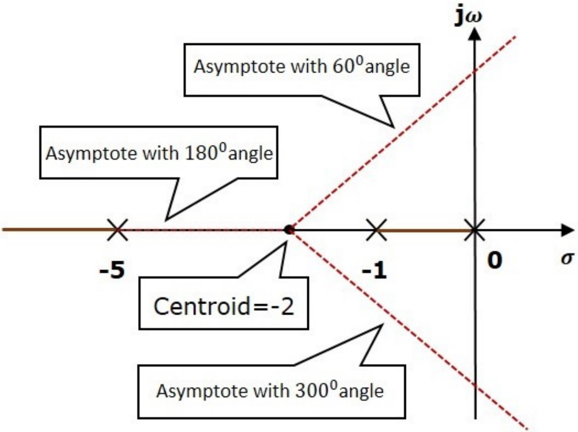
Centroid $\alpha = -2$

The angle of asymptotes are $\theta = 60^0, 180^0$ and 300^0 .

The centroid and three asymptotes are shown in the following figure.

Rules for Construction of Root Locus

The centroid and three asymptotes are shown in the following figure.



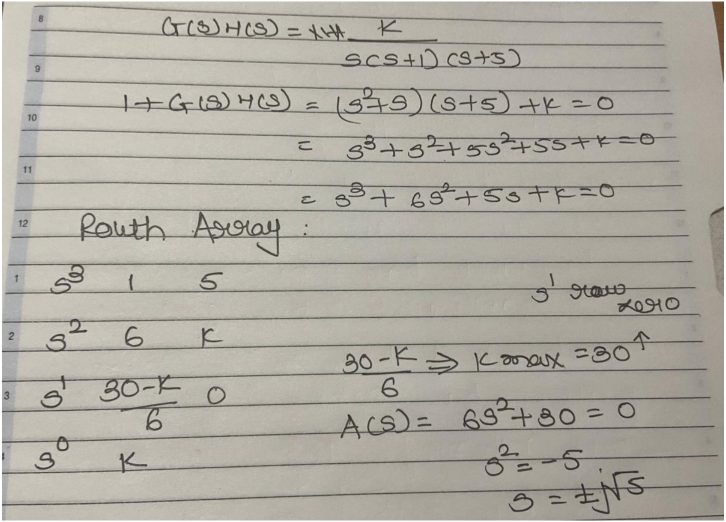
Rules for Construction of Root Locus

Rule 5 – Find the intersection points of root locus branches with an imaginary axis.

We can calculate the point at which the root locus branch intersects the imaginary axis and the value of **K** at that point by using the Routh array method and special **case (ii)**.

- If all elements of any row of the Routh array are zero, then the root locus branch intersects the imaginary axis and vice-versa.
- Identify the row in such a way that if we make the first element as zero, then the elements of the entire row are zero. Find the value of **K** for this combination.
- Substitute this **K** value in the auxiliary equation. You will get the intersection point of the root locus branch with an imaginary axis.

Rules for Construction of Root Locus



Rules for Construction of Root Locus

Step 3 – Since two asymptotes have the angles of 60^0 and 300^0, two root locus branches intersect the imaginary axis. By using the Routh array method and special case(ii), the root locus branches intersects the imaginary axis at $j\sqrt{5}$ and $-j\sqrt{5}$.

Rules for Construction of Root Locus

Rule 6 – Find Break-away and Break-in points.

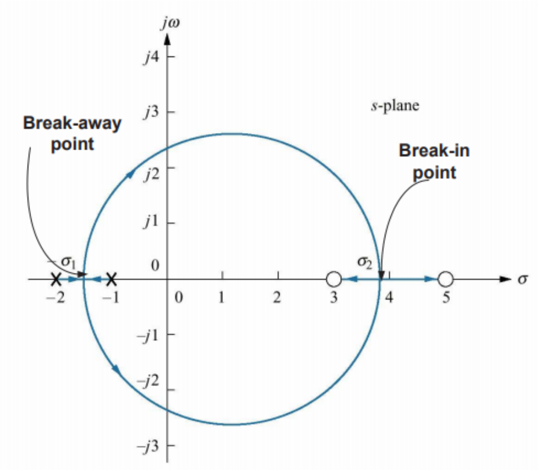
- If there exists a real axis root locus branch between two open loop poles, then there will be a **break-away point** in between these two open loop poles.
- If there exists a real axis root locus branch between two open loop zeros, then there will be a **break-in point** in between these two open loop zeros.

Note – Break-away and break-in points exist only on the real axis root locus branches.

Follow these steps to find break-away and break-in points.

- Write K in terms of s from the characteristic equation $1 + G(s)H(s) = 0$.
- Differentiate K with respect to s and make it equal to zero. Substitute these values of s in the above equation.
- The values of s for which the K value is positive are the **break points**.

Rules for Construction of Root Locus



Rules for Construction of Root Locus

$$1 + G(s)H(s) = 0$$

$$s^3 + 6s^2 + 5s + K = 0$$

$$K = -s^3 - 6s^2 - 5s$$

$$\frac{dK}{ds} = -3s^2 - 12s - 5 = 0$$

$$3s^2 + 12s + 5 = 0$$

Breakaway points $= \dfrac{-12 \pm \sqrt{144 - 60}}{2 \times 3}$

$= \dfrac{-12 \pm 9.17}{6}$

$= -3.52$ & -0.473

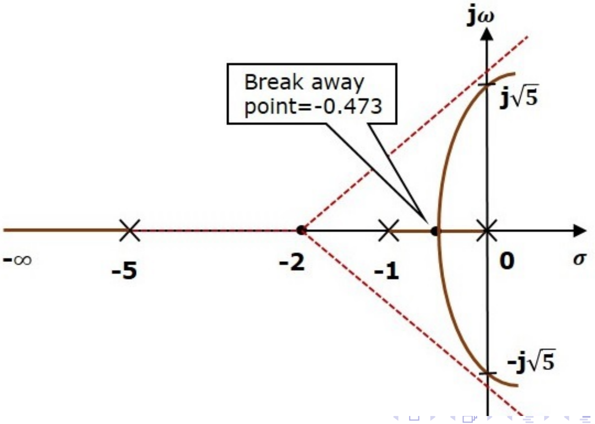
for $s = -3.52 \Rightarrow K =$ ~~43.61~~ -13.13

for $s = -0.473 \Rightarrow K = 1.131$.

Rules for Construction of Root Locus

There will be one break-away point on the real axis root locus branch between the poles $s = -1$ and $s = 0$. By following the procedure given for the calculation of break-away point, we will get it as $s = -0.473$.

The root locus diagram for the given control system is shown in the following figure.



Rules for Construction of Root Locus

Rule 7 – Find the angle of departure and the angle of arrival.

The Angle of departure and the angle of arrival can be calculated at complex conjugate open loop poles and complex conjugate open loop zeros respectively.

The formula for the **angle of departure** ϕ_d is

$$\phi_d = 180^0 - \phi$$

The formula for the **angle of arrival** ϕ_a is

$$\phi_a = 180^0 + \phi$$

Where,

$$\phi = \sum \phi_P - \sum \phi_Z$$

Rules for Construction of Root Locus

$$G(S)H(S) = \frac{K}{s(s+3)(s^2+3s+11.25)}$$

$$= \frac{K}{s(s+3)(s+1.5+j3)(s+1.5-j3)}$$

$$s = 0,\ 3,\ -1.5-j3,\ -1.5+j3$$

Rules for Construction of Root Locus

For angle of departure :

$$\phi_{P1} = 180^\circ - \tan^{-1}\frac{3}{1.5} = 116.56^\circ$$

$$\phi_{P2} = 90^\circ,\ \phi_{P3} = \tan^{-1}\frac{3}{1.5} = 63.43^\circ$$

$$\sum\phi_P = 270^\circ,\ \sum\phi_Z = 0^\circ$$

$$\phi = \sum\phi_P - \phi_Z = 270^\circ$$

$$\phi_d = 180^\circ - \phi = -90^\circ \text{ at } -1.5 + j3$$

and $+90^\circ$ at $-1.5 - j3$

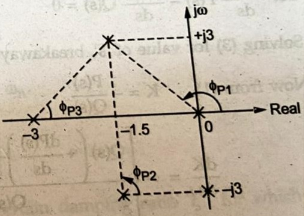


Fig. 9.5.12

...ints of the root locus are the solutions of $\frac{dK}{ds} =$

Example

Sketch the complete root locus of system having

$$G(s)H(s) = \frac{K}{s(s+1)(s+2)(s+3)}$$

Example Solution

Solution : Step 1 : P = 4, Z = 0, N = 4.

All branches approaching to ∞.

Starting points are s = 0, – 1, – 2, – 3.

Step 2 : Pole-Zero plot and sections of real axis are as shown in Fig. 9.7.3.

Sections between 0 and –1 and – 2 and – 3 are the parts of root locus.

According to general predictions minimum 2 breakaway points exists.

One between s = 0 and –1 and second between s = – 2 and s = – 3.

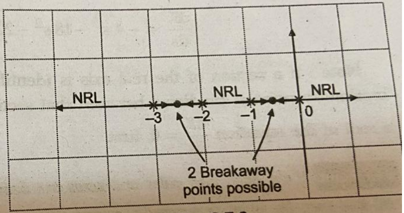


Fig. 9.7.3

Example Solution

Step 3 : Angles of asymptotes

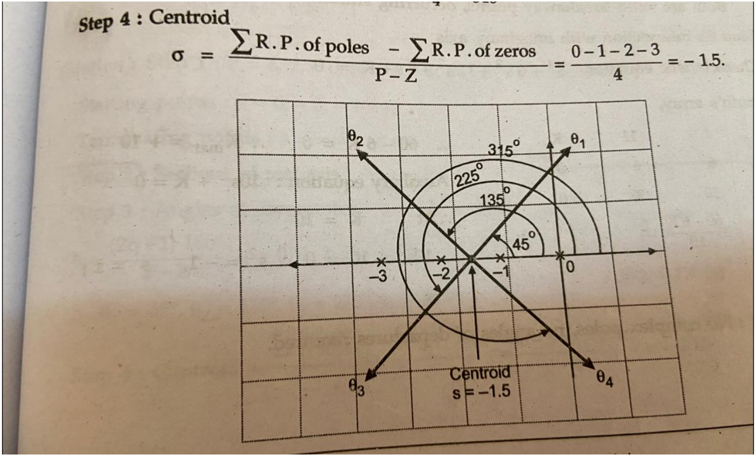
4 branches approaching to ∞ hence 4 asymptotes are required.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \quad q = 0, 1, 2, 3.$$

$q = 0$, $\theta_1 = 45^\circ$, $q = 1$, $\theta_2 = 135^\circ$

$q = 2$, $\theta_3 = 225^\circ$, $q = 3$, $\theta_4 = 315^\circ$

Example Solution



Step 4 : Centroid

$$\sigma = \frac{\sum R.P.\text{ of poles} - \sum R.P.\text{ of zeros}}{P - Z} = \frac{0-1-2-3}{4} = -1.5.$$

Example Solution

Step 5 : Breakaway point so characteristic equation is

$$1 + G(s)H(s) = 0 \quad \text{i.e.} \quad 1 + \frac{K}{s(s+1)(s+2)(s+3)} = 0$$

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0 \quad \text{i.e.} \quad K = -s^4 - 6s^3 - 11s^2 - 6s \qquad \ldots (1)$$

$$\therefore \quad \frac{dK}{ds} = -4s^3 - 18s^2 - 22s - 6 = 0 \quad \text{i.e.} \quad 4s^3 + 18s^2 + 22s + 6 = 0$$

Note : If a section of the real axis is identified for the existence of root locus and a breakaway point is predicted between that section. Try the midpoint of such section for a root of the equation $\frac{dK}{ds} = 0$, first.

Example Solution

Key Point *If the open loop poles and zeros are distributed symmetrically about a point on the real axis then the point of symmetry is one of the roots of equation* $\frac{dK}{ds} = 0$.

In this problem poles s = 0, – 1 and s = –2, –3 are symmetrically located about a point s = –1.5. So s = –1.5 must be one of the roots of dK/ds = 0

Solving $4s^3 + 18s^2 + 22s + 6 = 0$ we get s = – 1.5, – 0.381, – 2.619

But as there is no root locus between s = – 1 and s = – 2.

∴ s = – 1.5 cannot be valid breakaway point.

$$\left.\begin{array}{lll} \text{For} & s = -0.381, & K = 1 \\ \text{For} & s = 2.619, & K = 1 \end{array}\right\} \text{Substituting in equation (1)}$$

∴ **Both are valid breakaway points, occurring simultaneously at K = 1.**

Example Solution

Step 6 : Intersection with imaginary axis

Characteristic equation $s^4 + 6s^3 + 11s^2 + 6s + K = 0$

Routh's array,

s^4	1	11	K
s^3	6	6	0
s^2	10	K	0
s^1	$\frac{60-6K}{10}$	0	

$\therefore\ 60 - 6K = 0 \qquad \therefore\ K_{mar} = +10$

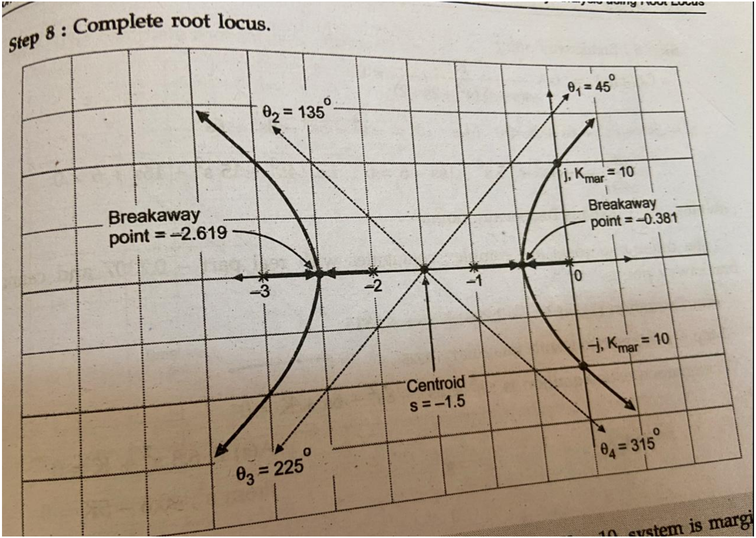
Auxiliary equation : $10s^2 + K = 0$

$\therefore$ At $K = 10$

$10s^2 + 10 = 0 \qquad s^2 = -1, \qquad s =$

Step 7 : No complex poles, no angles of departures required.

Example Solution



Example

Example1: Suppose we have given the transfer function of the closed system as:

$$G(s)H(s) = \frac{K}{s(s+5)(s+10)}$$

We have to construct the root locus for this system and predict the stability of the same.

Example Solution

Firstly, writing the characteristic equation of the above system,

$$s(s+5)(s+10) = 0$$

So, from the above equation, we get, s = 0, -5 and -10.

Thus, P = 3, Z = 0 and since P > Z therefore, the number of branches will be equal to the number of poles.

So, N = P = 3

Thus, under this condition, the branches will start from the locations of 0, -5 and -10 in the s-plane and will approach infinity.

Example Solution

Now, let us calculate the angle of asymptotes with the formula given below:

$$\theta = \frac{(2q + 1)180^\circ}{P - Z}$$

: *q lies between 0 to P-Z-1*

So, in this case, θ will be calculated for q = 0, 1 and 2.

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ$$

$$\theta_2 = \frac{3 * 180^\circ}{3} = 180^\circ$$

$$\theta_3 = \frac{5 * 180^\circ}{3} = 300^\circ$$

So, these three are the angle possessed by asymptotes approaching infinity.

Example Solution

Now, let us check where the centroid lies on the real axis by using the formula given below:

$$\sigma = \frac{\text{sum of real part of poles - sum of real part of zeros}}{P-Z}$$

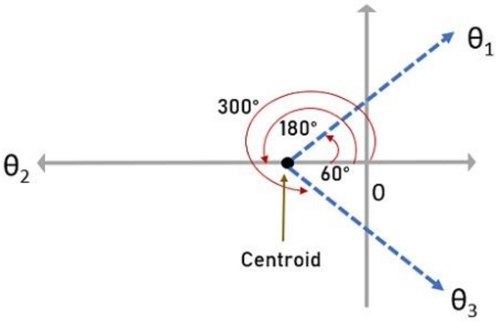
$$\sigma = \frac{0-5-10-0}{3}$$

$$\sigma = \frac{-15}{3}$$

$$\sigma = -5$$

Example Solution

The figure below represents a rough sketch of the plot that is obtained by the above analysis



Example Solution

Earlier we have predicted that one breakaway point will be present in the section between points 0 and -5. So, now using the method to determine the breakaway point we will check the validity of the breakaway point.

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s+5)(s+10)} = 0$$

$$s(s+5)(s+10) + K = 0$$

$$s^3 + 15s^2 + 50s + K = 0$$

$$K = - s^3 - 15s^2 - 50s$$

In this method, roots obtained on differentiating K with respect to s and equating it to 0, will be the breakaway point.

Example Solution

Therefore,

$$\frac{dK}{ds} = 0$$

$$\frac{d(-s^3 - 15s^2 - 50s)}{ds} = 0$$

$$-3s^2 - 30s - 50 = 0$$

Or

$$3s^2 + 30s + 50 = 0$$

Thus, on solving, roots obtained will be **-2.113** and **-7.88**.

As the root -7.88 falls beyond the predicted section for the breakaway point thus s = -2.113 is the valid breakaway point.

Example Solution

Further, we can get the value of K on substituting the value of s= -2.113 in the equation,

$$K = (-2.113)^3 - 15(-2.113)^2 - 50(-2.113)$$

Therefore, on solving,

$$K = 48.11$$

Here, K obtained is a positive value, hence, **s = -2.113** is valid.

Now, we have to check the at what point the root locus intersects with the imaginary axis. Thus, for this routh array is used.

Here a proper method is used where the characteristic equation is used and routh array in terms of K is formed.

$$s^3 + 15s^2 + 50s + K = 0$$

Example Solution

Thus, the Routh's Array:

s^3	1	50
s^2	15	K
s^1	$\frac{750 - K}{15}$	0
s^0	K	

Now, to find K_{mar}, which is the value of K from one of the rows of routh's array as row of zeros, except the row s^0.

Considering, row s^1, 750 – K = 0

Thus, $\mathbf{K_m = 750}$

Example Solution

Further, with the help of coefficients of the row which is present above the row of zero, an auxiliary equation A(s) = 0 is constructed. In this case,

$$A(s) = 15s^2 + K = 0$$

So, substituting the value of Km in the above equation, we will get,

$$15s^2 + 750 = 0$$

$$s^2 = -\frac{750}{15} = -50$$

Therefore,

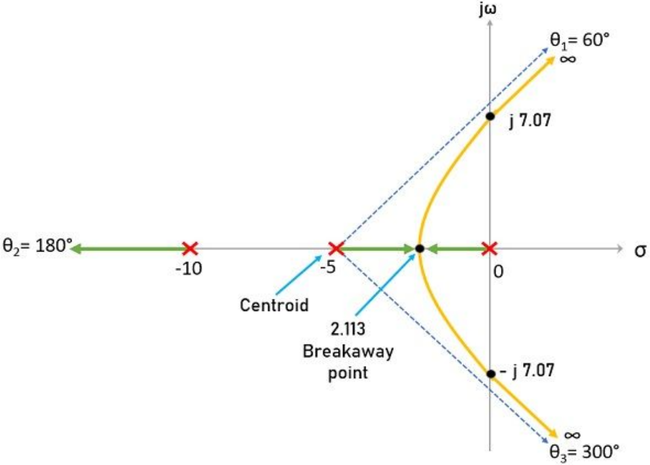
$$s = \pm j\sqrt{50}$$

$$s = + j\,7.07, -j\,7.07$$

Thus, these are the intersection points of the root locus with the imaginary axis.

Also, as the poles are not complex thus angles of departure not needed. Hence, at the breakaway point, the root locus breaks at ± 90°.

Example Solution



Example

Example2: Consider that for the system with transfer function given below we have to sketch the root locus and predict its stability.

$$G(s)H(s) = \frac{K}{s(s^2 + 2s + 2)}$$

Example Solution

The characteristic equation provides the poles and zeros. So, writing the characteristic equation for the above system:

$$s(s^2 + 2s + 2) = 0$$

Thus, s = 0,

$$s = \frac{-2 \pm \sqrt{4-8}}{2}$$

On solving

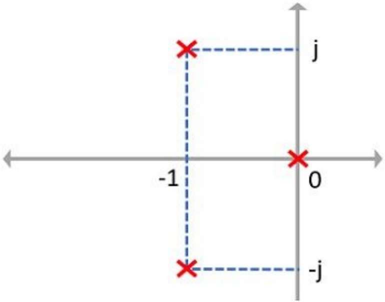
$$\mathbf{s = -1 + j \text{ and } -1 - j}$$

Thus, here

P = 3, Z = 0 and as P > Z, so rule wise N = P = 3

Example

The pole-zero plot is given below:



Example Solution

Here, it is clear that branch originating from s = 0 approaches -∞. And general predictions clear that there is no breakaway point here.

Angle of asymptotes

Since

$$\theta = \frac{(2q+1)180°}{P-Z}$$

: *q lies between 0 to P-Z-1*

So, here, θ will be calculated for q = 0, 1 and 2.

$$\theta_1 = \frac{180°}{3} = 60°$$

$$\theta_2 = \frac{3*180°}{3} = 180°$$

$$\theta_3 = \frac{5*180°}{3} = 300°$$

Example

Now, centroid:

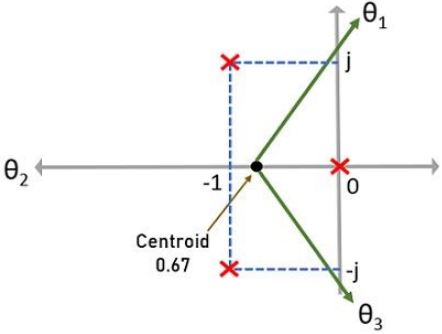
$$\sigma = \frac{\text{sum of real part of poles} - \text{sum of real part of zeros}}{P-Z}$$

$$\sigma = \frac{0-1-1-0}{3}$$

$$\sigma = \frac{-2}{3} = -0.67$$

Example Solution

So, with the help of the above analysis, the sketch of s-plane is given below:



Example

Now, let us check for the breakaway point.

$$1 + G(s)H(s) = 0$$

$$1 + \frac{K}{s(s^2 + 2s + 2)} = 0$$

$$s^3 + 2s^2 + 2s + K = 0$$

$$K = -s^3 - 2s^2 - 2s$$

So, on differentiating,

$$\frac{dK}{ds} = 0$$

$$\frac{d(-s^3 - 2s^2 - 2s)}{ds} = 0$$

$$-3s^2 - 4s - 4 = 0$$

Example Solution

Or

$$3s^2 + 4s + 4 = 0$$

Breakaway points are calculated as:

$$s = \frac{-4 \pm \sqrt{16-24}}{6}$$

Therefore,

$$\mathbf{s = -0.67 \pm j\,0.47}$$

Now, as here we are having complex conjugates, thus, checking the validity of these points as breakaway point by using angle condition.

Testing, s = – 0.67 + j 0.47

$$\angle G(s)H(s) = \pm (2q + 1)180°$$

$$: q = 0, 1, 2$$

$$G(s)H(s) = \frac{K}{s(s+1-j)(s+1+j)}$$

Example

At s = – 0.67 + j 0.47

$$\angle G(s)H(s) = \frac{\angle K+j0}{\angle -0.67+j0.47 \angle -0.67+j0.47+1-j \angle -0.67+j0.47+1+j}$$

On solving,

$$\angle G(s)H(s) = -164.11°$$

As it is not an odd multiple of 180° thus, this point is not present on the root locus, hence there is no breakaway point here.

Example Solution

Further, checking for the intersection with the imaginary axis.

$$s^3 + 2s^2 + 2s + K = 0$$

Routh's array

s^3	1	2
s^2	2	K
s^1	$\frac{4-K}{2}$	0
s^0	-K	

The value of $K_m = +4$ makes $s^1 = 0$

Example

Thus,

$$A(s) = 2s^2 + K = 0$$

Substituting $K_m = 4$, we get,

$$2s^2 + 4 = 0$$

$$s = \pm j\,1.414$$

Now, calculating angle of departure

At complex pole, **– 1 + j**

$$\varphi_d = 180° - \varphi$$

So, here $\varphi_{P1} = 135°$ and $\varphi_{P2} = 90°$

$$\varphi_d = 180° - (135° + 90°)$$

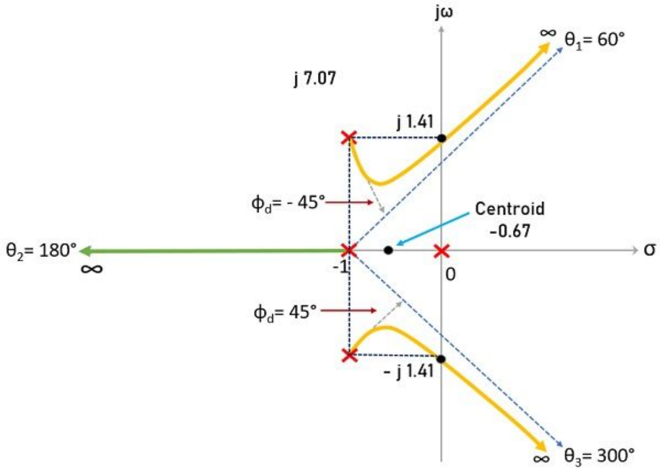
$$\varphi_d = -45°$$

Therefore, at the complex pole, **– 1 – j**,

$$\varphi_d = +45°$$

Example Solution

So, with the above-determined values and parameters, the complete root locus sketch obtained is given below:



Example Solution

Now talking about stability, for **K between 0 to 4**, the roots are present on the left half of s-plane, representing a **completely stable system**.

K = +4 makes the system **marginally stable** due to the presence of dominant roots on the imaginary axis. While for **K > 4**, the system becomes **unstable** as dominant roots lie in the right half of s-plane.

Example

Example 9.7.5 *A feedback control system has open loop transfer function*

$$G(s)H(s) = \frac{K}{s(s+4)(s^2+4s+20)}$$

Plot the root locus for K = 0 to ∞. Indicate all the points on it.

Feb.-04,

Solution Example

Solution : The open loop poles are located at

$$s = 0, -4 \text{ and } s = \frac{-4 \pm \sqrt{16-80}}{2} = -2 \pm j\,4$$

Step 1 : Initial data : P = 4, Z = 0,

N = P = 4 branches,

$$P - Z = 4 \text{ approaching to } \infty.$$

Starting points = $0, -4, -2 + j\,4, -2 - j\,4$

Terminating points = $\infty, \infty, \infty, \infty$.

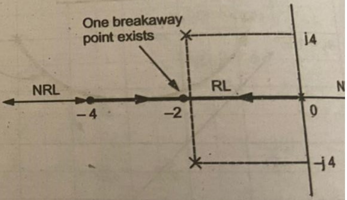


Fig. 9.7.8 (a)

Step 2 : Section of real axis as shown in the Fig. 9.7.8 (a).

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)\,180^\circ}{P-Z}, \; q = 0, 1, 2, 3$$

Solution Example

Step 3 : Angles of asymptotes

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \; q = 0, 1, 2, 3$$

$$\theta_1 = \frac{180^\circ}{4} = 45^\circ \qquad \theta_2 = \frac{3\times180^\circ}{4} = 135^\circ$$

$$\theta_3 = \frac{5\times180^\circ}{4} = 225^\circ \qquad \theta_4 = \frac{7\times180^\circ}{4} = 315^\circ$$

Step 4 : Centroid

$$\sigma = \frac{\sum \text{R.P. of poles} - \sum \text{R. P. of zeros}}{P-Z} = \frac{(0-4-2-2)-(0)}{4} = -2$$

Solution Example

Step 5 : Breakaway points

$$1 + G(s)H(s) = 0 \text{ i.e. } 1+\frac{K}{s(s+4)\left(s^2+4s+20\right)} = 0$$

$$\therefore s^4 + 8s^3 + 36s^2 + 80s + K = 0 \quad \text{i.e.} \quad K = -s^4 - 8s^3 - 36s^2 - 80s \quad \text{... (1)}$$

$$\frac{dK}{ds} = -4s^3 - 24s^2 - 72s - 80 \quad \text{i.e.} \quad s^3 + 6s^2 + 18s + 20 = 0$$

Solving, $s = -2$ and $-2 \pm j2.45$

All are valid breakaway points. The validity of $-2 \pm j\,2.45$ as a breakaway point can be confirmed by using angle condition. (Refer example 9.5.9)

At $s = -2$, $K = 64$, from equation (1), hence $s = -2$ is valid.

Solution Example

Step 6 : Intersection with imaginary axis

The characteristic equation is already obtained as,

$$s^4 + 8s^3 + 36s^2 + 80s + K = 0$$

The Routh's array is,

s^4	1	36	K
s^3	8	80	0
s^2	26	K	
s^1	$\frac{2080 - 8K}{26}$	0	
s^0	K		

$\therefore \quad 2080 - 8K = 0$

$\therefore \quad K_{mar} = \frac{2080}{8} = 260$

$A(s) = 26s^2 + K = 0$

$\therefore \quad 26s^2 + 260 = 0$

$\therefore \quad s^2 = -10$

$\therefore \quad s = \pm j\sqrt{10} = \pm j\,3.162$

Solution Example

imaginary axis. ... points with

Step 7 : Angle of departure

$\phi_{P1} = 180° - x = 180° - \tan^{-1}\left(\frac{4}{2}\right)$

$= 180° - 63.43° = +116.56°$

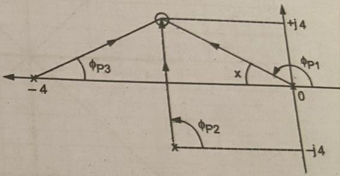


Fig. 9.7.8 (b)

$\phi_{P2} = +90°$ and $\phi_{P3} = \tan^{-1}\left(\frac{4}{2}\right)$

$= +63.43°$

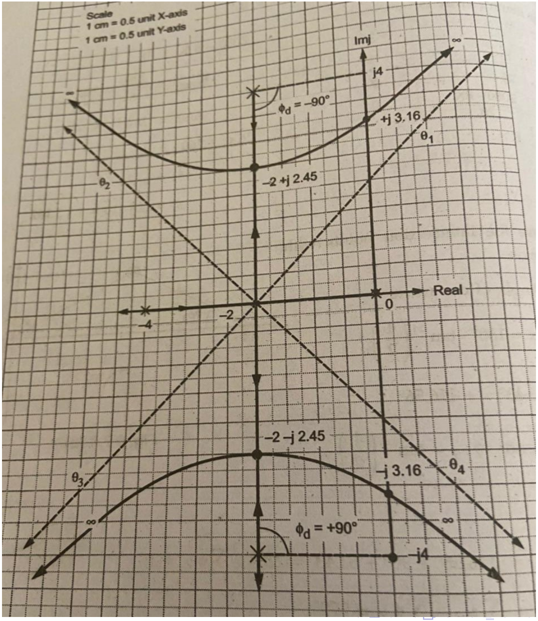
$\therefore \quad \sum\phi_P = 116.56° + 90° + 63.43° = 270°, \quad \sum\phi_Z = 0°$

$\therefore \quad \phi = \sum\phi_P - \sum\phi_Z = 270°$

$\therefore \quad \phi_d = 180° - \phi = 180° - 270° = -90°$ at $-2 + j4$

While ϕ_d at $-2 - j4$ is, $\phi_d = +90°$

Solution Example



Example

Example 9.7.14 Given : $G(s) = \frac{K(s+1)}{(s+2)(s+3)(s+4)}$ for a negative unity feedback system.

i)Sketch the root locus with necessary calculations. Show at least one TEST POINT on the complex plane on the root locus, where criterion is satisfied.

ii)If $K = 10$, where are the roots ?

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Solution Example

Solution : $G(s) = \frac{K(s+1)}{(s+2)(s+3)(s+4)}$, $H(s) = 1$

Step 1 : P = 3, Z = 1, P − Z = 2 branches approach to ∞

Starting points : − 2, − 3, − 4

Terminating points : − 1, ∞, ∞

Step 2 : Sections of real axis.

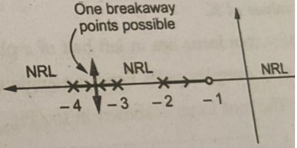


Fig. 9.7.15

Step 3 : Two asymptotes.

$$\theta = \frac{(2q+1)180^\circ}{P-Z}, \quad q = 0, 1$$

$$\theta_1 = 90^\circ, \qquad \theta_2 = 270^\circ$$

$\dots$ O.L. poles $- \sum$ R.P. of O.L. zeros $= \frac{-2-3-4-\dots}{2}$

Solution Example

Step 4 : Centroid $= \dfrac{\sum \text{R.P. of O.L. poles} - \sum \text{R.P. of O.L. zeros}}{P - Z} = \dfrac{-2-3-4-(-1)}{2}$

$= -4$

Step 5 : Breakaway point

$$1 + G(s)H(s) = 1 + \frac{K(s+1)}{(s+2)(s+3)(s+4)} = 0$$

$$(s+2)(s+3)(s+4) + K(s+1) = 0$$

$$K = \frac{-(s+2)(s+3)(s+4)}{(s+1)} = \frac{-s^3 - 9s^2 - 26s - 24}{(s+1)} \quad \ldots (1)$$

$$\frac{dK}{ds} = \frac{(s+1)(-3s^2 - 18s - 26) - (-s^3 - 9s^2 - 26s - 24)(1)}{(s+1)^2} = 0$$

$$\therefore -3s^3 - 3s^2 - 18s^2 - 18s - 26s - 26 + s^3 + 9s^2 + 26s + 24 = 0$$

$$\therefore 2s^3 + 12s^2 + 18s + 2 = 0$$

Solving, $s = -0.12, -3.532, -2.3472$

Only $s = -3.532$ is a valid breakaway point for which $K = +0.1509$ from the equation (1).

Solution Example

Step 6 : Intersection with imaginary axis.

The characteristic equation is,

$$s^3 + 9s^2 + s(26+K) + (24+K) = 0$$

s^3	1	$26 + K$
s^2	9	$24 + K$
s^1	$\frac{234 + 9K - 24 - K}{9}$	0

From row of s^1,

$$120 + 8K = 0$$

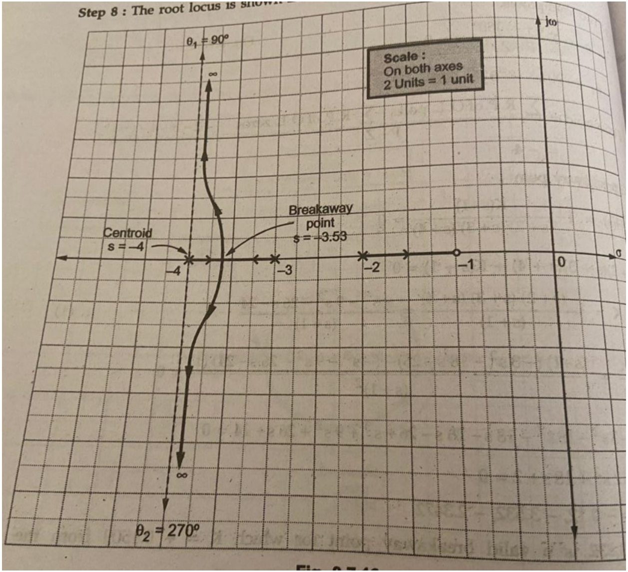
$$\therefore \quad K_{mar} = -26.25$$

As K_{mar} is negative there is no intersection of root locus with the imaginary axis for positive values of K.

The entire root locus lies in left half of s plane.

Step 7 : No complex poles hence no angle of departure.

Solution Example




THANK YOU
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