Control Systems UNIT 3 Stability analysis

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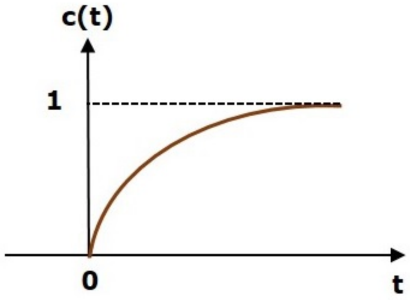
What is Stability?

- A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

Response of a stable system

This is the response of first order control system for unit step input. This response has the values between 0 and 1. So, it is bounded output.



We know that the unit step signal has the value of one for all positive values of t including zero. So, it is bounded input. Therefore, the first order control system is stable since both the input and the output are bounded.

Types of Systems based on Stability

We can classify the systems based on stability as follows:

- Absolutely stable system
- Conditionally stable system
- Marginally stable system

Absolutely Stable System

- If the system is stable for all the range of system component values, then it is known as the absolutely stable system.
- The open loop control system is absolutely stable if all the poles of the open loop transfer function present in left half of s plane.
- Similarly, the closed loop control system is absolutely stable if all the poles of the closed loop transfer function present in the left half of the s plane.

Conditionally Stable System

If the system is stable for a certain range of system component values, then it is known as conditionally stable system.

Marginally Stable System

- If the system is stable by producing an output signal with constant amplitude and constant frequency of oscillations for bounded input, then it is known as marginally stable system.
- The open loop control system is marginally stable if any two poles of the open loop transfer function is present on the imaginary axis.
- Similarly, the closed loop control system is marginally stable if any two poles of the closed loop transfer function is present on the imaginary axis.

Routh-Hurwitz Stability Criterion

- Routh-Hurwitz stability criterion is having one necessary condition and one sufficient condition for stability.
- If any control system doesnt satisfy the necessary condition, then we can say that the control system is unstable. But, if the control system satisfies the necessary condition, then it may or may not be stable.
- So, the sufficient condition is helpful for knowing whether the control system is stable or not.

Necessary Condition for Routh-Hurwitz Stability

The necessary condition is that the coefficients of the characteristic polynomial should be positive. This implies that all the roots of the characteristic equation should have negative real parts.

Consider the characteristic equation of the order 'n' is -

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0 = 0$$

Note that, there should not be any term missing in the **n**$^{\mathbf{th}}$ order characteristic equation. This means that the **n**$^{\mathbf{th}}$ order characteristic equation should not have any coefficient that is of zero value.

Sufficient Condition for Routh-Hurwitz Stability

The sufficient condition is that all the elements of the first column of the Routh array should have the same sign. This means that all the elements of the first column of the Routh array should be either positive or negative.

Routh Array Method

- If all the roots of the characteristic equation exist to the left half of the s plane, then the control system is stable. If at least one root of the characteristic equation exists to the right half of the s plane, then the control system is unstable. So, we have to find the roots of the characteristic equation to know whether the control system is stable or unstable. But, it is difficult to find the roots of the characteristic equation as order increases.
- So, to overcome this problem there we have the Routh array method. In this method, there is no need to calculate the roots of the characteristic equation. First formulate the Routh table and find the number of the sign changes in the first column of the Routh table. The number of sign changes in the first column of the Routh table gives the number of roots of characteristic equation that exist in the right half of the s plane and the control system is unstable.

Procedure

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \ldots + a_{n-1} s^1 + a_n s^0$$

s^n	a_0	a_2	a_4	a_6	$\cdots$	$\cdots$
s^{n-1}	a_1	a_3	a_5	a_7	$\cdots$	$\cdots$
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	$b_3 = \frac{a_1 a_6 - a_7 a_0}{a_1}$	$\cdots$	$\cdots$	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$c_2 = \frac{b_1 a_5 5 - b_3 a_1}{b_1}$	$\vdots$			
$\vdots$	$\vdots$	$\vdots$	$\vdots$			
s^1	$\vdots$	$\vdots$				
s^0	a_n					

Example

Let us find the stability of the control system having characteristic equation,

$$s^4 + 3s^3 + 3s^2 + 2s + 1 = 0$$

Solution Example

Step 1 – Verify the necessary condition for the Routh-Hurwitz stability.

All the coefficients of the characteristic polynomial, $s^4 + 3s^3 + 3s^2 + 2s + 1$ are positive. So, the control system satisfies the necessary condition.

Solution Example

Step 2 – Form the Routh array for the given characteristic polynomial.

s^4	1	3	1
s^3	3	2	
s^2	$\frac{(3\times3)-(2\times1)}{3}=\frac{7}{3}$	$\frac{(3\times1)-(0\times1)}{3}=\frac{3}{3}$ $=1$	
s^1	$\frac{\left(\frac{7}{3}\times2\right)-(1\times3)}{\frac{7}{3}}=\frac{5}{7}$		
s^0	1		

Solution Example

Step 3 Verify the sufficient condition for the Routh-Hurwitz stability.

All the elements of the first column of the Routh array are positive. There is no sign change in the first column of the Routh array. So, the control system is stable.

Example

Examine stability of $F(s) = s^3 + 6s^2 + 11s + 6 = 0$

Solution Example

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

$$
\begin{array}{c|cc}
s^3 & 1 & 11 \\
s^2 & 6 & 6 \\
s^1 & \dfrac{11\times 6-6}{6} = 10 & 0 \\
s^0 & 6 &
\end{array}
$$

As there is no sign change in first column, system is stable.

Example

Comment on the stability of a system using Routh's stability criteria whose characteristic equation is

$$F(s) = s^4 + 2s^3 + 4s^2 + 6s + 8 = 0$$

How many poles of systems lie in right half of s-plane?

Solution Example

Solution : Routh's array is,

s^4	1	4	8
s^3	2	6	0
s^2	$+1$ $\downarrow$	8	0
s^1	-10 $\downarrow$	0	
s^0	$+8$		

There are two sign changes in the first column hence system is **unstable** with **two** poles in right half of s-plane.

Special Cases of Routh Array

We may come across two types of situations, while forming the Routh table. It is diffic
complete the Routh table from these two situations.

The two special cases are –

- The first element of any row of the Routh array is zero.
- All the elements of any row of the Routh array are zero.

First Element of any row of the Routh array is zero

If any row of the Routh array contains only the first element as zero and at least one of the remaining elements have non-zero value, then replace the first element with a small positive integer, ϵ. And then continue the process of completing the Routh table. Now, find the number of sign changes in the first column of the Routh table by substituting ϵ tends to zero.

Example

Let us find the stability of the control system having characteristic equation,

$$F(s) = s^4 + 2s^3 + s^2 + 2s + 1 = 0$$

Solution Example

s^4	1	1	1
s^3	$\cancel{2}$ 1	$\cancel{2}$ 1	
s^2	$\frac{(1\times1)-(1\times1)}{1} = 0$	$\frac{(1\times1)-(0\times1)}{1} = 1$	
s^1			
s^0			

Solution Example

s^4	1	1	1
s^3	1	1	
s^2	ϵ	1	
s^1	$\frac{(\epsilon\times1)-(1\times1)}{\epsilon} = \frac{\epsilon-1}{\epsilon}$		
s^0	1		

Solution Example

As ϵ tends to zero, the Routh table becomes like this.

s^4	1	1	1
s^3	1	1	
s^2	0	1	
s^1	$-\infty$		
s^0	1		

Example

Let us find the stability of the control system having characteristic equation,

$$F(s) = s^5 + 2s^4 + 3s^3 + 4s^2 + 5s + 6 = 0$$

How many poles lie in right half of s-plane?

Solution Example

Solution : The Routh's array is,

s^5	1	3	5
s^4	2	4	6
s^3	1	2	0
s^2	$0+\varepsilon$	6	
s^1	$\frac{2\varepsilon-6}{\varepsilon}$	0	
s^0	6		

$\Leftarrow$ Special case 1

$$\lim_{\varepsilon \to 0} \frac{2\varepsilon-6}{\varepsilon} \to -\text{Ve}$$

Thus there are two sign changes in the first column hence two poles lie in right half of s-plane.

System unstable.

All the Elements of any row of the Routh array are zero

In this case, follow these two steps –

- Write the auxilary equation, A(s) of the row, which is just above the row of zeros.
- Differentiate the auxiliary equation, A(s) with respect to s. Fill the row of zeros with these coefficients.

Example

Let us find the stability of the control system having characteristic equation,

$$F(s) = s^5 + 3s^4 + s^3 + 3s^2 + s + 3 = 0$$

Solution

Routh Array

s^5	1	1	1
s^4	~~3~~ 1	~~3~~ 1	~~3~~ 1
s^3	$\frac{(1\times1)-(1\times1)}{1}=0$	$\frac{(1\times1)-(1\times1)}{1}=0$	
s^2			
s^1			
s^0			

Solution

Routh Array

Special case (ii) – All the elements of row s^3 are zero. So, write the auxiliary equation, A(s) of the row s^4 .

$$A(s) = s^4 + s^2 + 1$$

Differentiate the above equation with respect to s.

$$\frac{\mathrm{d}A(s)}{\mathrm{d}s} = 4s^3 + 2s$$

Solution

Place these coefficients in row s^3

s^5	1	1	1
s^4	1	1	1
s^3	~~4~~ 2	~~2~~ 1	
s^2	$\frac{(2\times1)-(1\times1)}{2} = 0.5$	$\frac{(2\times1)-(0\times1)}{2} = 1$	
s^1	$\frac{(0.5\times1)-(1\times2)}{0.5} = \frac{-1.5}{0.5} = -3$		
s^0	1		

Solution

There are two sign changes in the first column of Routh table. Hence, the control system is unstable.

Example

Let us find the stability of the control system having characteristic equation,

$$F(s) = s^6 + 2s^5 + 8s^4 + 12s^3 + 20s^2 + 16s + 16 = 0$$

Solution

Solution :

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	0	0	0	0

Row of zeros

Solution

$$A(s) = 2s^4 + 12s^2 + 16 = 0$$

$$\frac{dA}{ds} = 8s^3 + 24s = 0$$

s^6	1	8	20	16
s^5	2	12	16	0
s^4	2	12	16	0
s^3	8	24	0	0
s^2	6	16	0	
s^1	2.67	0		
s^0	16			

Solution

No sign change, so system may be stable. But as there is row of zero, system will be (i) marginally stable or (ii) unstable. To examine this solve A(s) = 0.

$$2s^4 + 12s^2 + 16 = 0$$

$$s^4 + 6s^2 + 8 = 0$$

Put $s^2 = y$

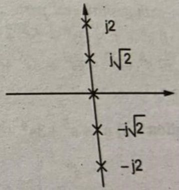
$$\therefore \quad y^2 + 6y + 8 = 0$$

$$y = \frac{-6 \pm \sqrt{36-32}}{2}$$

$$= -3 \pm 1 = -2, -4$$

$$\therefore \quad s^2 = -2 \quad \text{and} \quad s^2 = -4$$

$$\therefore \quad s = \pm j\sqrt{2} \quad \text{and} \quad s = \pm j2$$



Nonrepeated roots on imaginary axis. Hence system is marginally stable.

Example

Determine the range of K for stability for the following closed-loop transfer function

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

Solution

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

$$\begin{array}{llll} s^4 & 1 & 3 & K \\ s^3 & 3 & 2 & 0 \\ s^2 & \frac{7}{3} & K & \\ s^1 & 2 - \frac{9}{7}K & & \\ s^0 & K & & \end{array}$$

Solution

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.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