Control Systems UNIT 2

Ripal Patel

Assistant Professor,
Dr.Ambedkar Institute of Technology, Bangalore.

ripal.patel.ec@drait.edu.in

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Time response of control systems

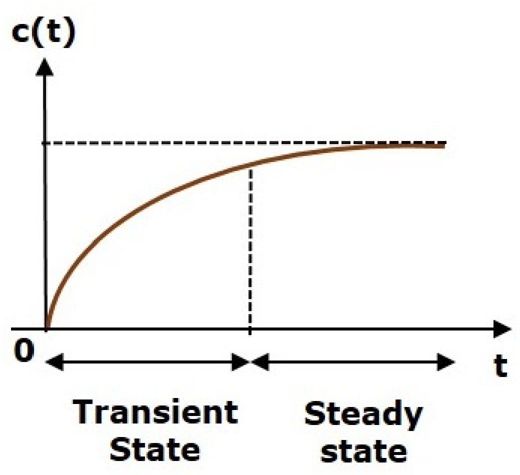
- We can analyze the response of the control systems in both the time domain and the frequency domain.

What is Time Response?

If the output of control system for an input varies with respect to time, then it is called the time response of the control system. The time response consists of two parts.

- Transient response
- Steady state response

Time response of control systems



Time response of control systems

Here, both the transient and the steady states are indicated in the figure. The responses corresponding to these states are known as transient and steady state responses.

Mathematically, we can write the time response c(t) as

$$c(t) = c_{tr}(t) + c_{ss}(t)$$

Where,

$c_{tr}(t)$ is the transient response

$c_{ss}(t)$ is the steady state response

Transient Response

- After applying input to the control system, output takes certain time to reach steady state. So, the output will be in transient state till it goes to a steady state. Therefore, the response of the control system during the transient state is known as transient response.
- The transient response will be zero for large values of t. Ideally, this value of t is infinity and practically, it is five times constant.
- Mathematically, we can write it as

$$\lim_{t \to \infty} c_{tr}(t) = 0$$

Steady State Response

- The part of the time response that remains even after the transient response has zero value for large values of t is known as steady state response. This means, the transient response will be zero even during the steady state.
- Example:
  Let us find the transient and steady state terms of the time response of the control system $c(t) = 10 + 5e^{-t}$
  Here, the second term $5e^{-t}$ will be zero as t denotes infinity. So, this is the transient term. And the first term 10 remains even as t approaches infinity. So, this is the steady state term.

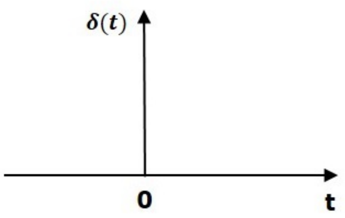
Unit Impulse Signal

A unit impulse signal, δ(t) is defined as

$$\delta(t) = 0 \quad \text{for} \quad t \neq 0$$

$$\text{and} \quad \int_{0^-}^{0^+} \delta(t)dt = 1$$

The following figure shows unit impulse signal.



So, the unit impulse signal exists only at 't' is equal to zero. The area of this signal under small interval of time around 't' is equal to zero is one. The value of unit impulse signal is zero for all other values of 't'.

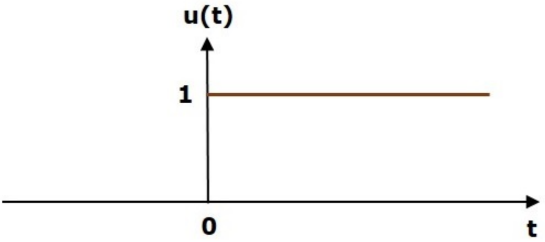
Unit Step Signal

A unit step signal, u(t) is defined as

$$u(t) = 1; t \geq 0$$

$$= 0; t < 0$$

Following figure shows unit step signal.



So, the unit step signal exists for all positive values of 't' including zero. And its value is one during this interval. The value of the unit step signal is zero for all negative values of 't'.

Unit Ramp Signal

A unit ramp signal, r(t) is defined as

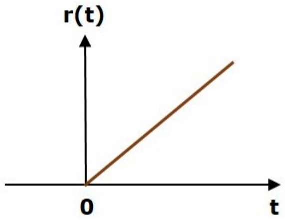
$$r(t) = t; t \geq 0$$

$$= 0; t < 0$$

We can write unit ramp signal, $r(t)$ in terms of unit step signal, $u(t)$ as

$$r(t) = tu(t)$$

Following figure shows unit ramp signal.



Unit Parabolic Signal

A unit parabolic signal, p(t) is defined as,

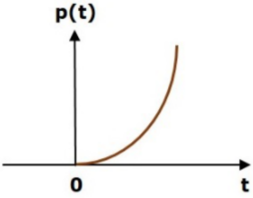
$$p(t) = \frac{t^2}{2}; t \geq 0$$

$$= 0; t < 0$$

We can write unit parabolic signal, $p(t)$ in terms of the unit step signal, $u(t)$ as,

$$p(t) = \frac{t^2}{2} u(t)$$

The following figure shows the unit parabolic signal.

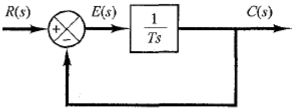


Stability

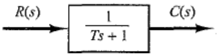
- A linear time-invariant control system is stable if the output eventually comes back to its equilibrium state when the system is subjected to an initial condition.
- A linear time-invariant control system is critically stable if oscillations of the output continue forever.
- It is unstable if the output diverges without bound from its equilibrium state when the system is subjected to an initial condition.

First Order System

Physically, RC circuit, thermal system etc.



(a)



(b)

First Order System

We know that the transfer function of the closed loop control system has unity negative feedback as,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

Substitute, $G(s) = \frac{1}{sT}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\frac{1}{sT}}{1+\frac{1}{sT}} = \frac{1}{sT+1}$$

First Order System

The power of s is one in the denominator term. Hence, the above transfer function is of the first order and the system is said to be the **first order system**.

We can re-write the above equation as

$$C(s) = \left(\frac{1}{sT+1}\right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal c(t),
- **R(s)** is the Laplace transform of the input signal r(t), and
- **T** is the time constant.

Impulse Response of First Order System

Consider the **unit impulse signal** as an input to the first order system.

So, $r(t) = \delta(t)$

Apply Laplace transform on both the sides.

$$R(s) = 1$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = 1$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)(1) = \frac{1}{sT+1}$$

Impulse Response of First Order System

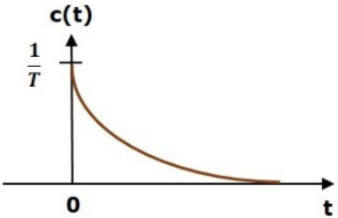
Rearrange the above equation in one of the standard forms of Laplace transforms.

$$C(s) = \frac{1}{T\left(s + \frac{1}{T}\right)} \Rightarrow C(s) = \frac{1}{T}\left(\frac{1}{s + \frac{1}{T}}\right)$$

Apply inverse Laplace transform on both sides.

$$c(t) = \frac{1}{T} e^{\left(-\frac{t}{T}\right)} u(t)$$

The unit impulse response is shown in the following figure.



Step Response of First Order System

Consider the **unit step signal** as an input to first order system.

$$\text{So,} \quad r(t) = u(t)$$

Apply Laplace transform on both the sides.

$$R(s) = \frac{1}{s}$$

Consider the equation, $C(s) = \left(\frac{1}{sT+1}\right) R(s)$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{1}{sT+1}\right)\left(\frac{1}{s}\right) = \frac{1}{s\,(sT+1)}$$

Step Response of First Order System

Do partial fractions of C(s).

$$C(s) = \frac{1}{s\left(sT+1\right)} = \frac{A}{s} + \frac{B}{sT+1}$$

$$\Rightarrow \frac{1}{s\left(sT+1\right)} = \frac{A\left(sT+1\right)+Bs}{s\left(sT+1\right)}$$

On both the sides, the denominator term is the same. So, they will get cancelled by each other. Hence, equate the numerator terms.

$$1 = A\left(sT+1\right)+Bs$$

By equating the constant terms on both the sides, you will get A = 1.

Substitute, A = 1 and equate the coefficient of the **s** terms on both the sides.

$$0 = T + B \Rightarrow B = -T$$

Step Response of First Order System

Substitute, A = 1 and B = −T in partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{T}{sT+1} = \frac{1}{s} - \frac{T}{T\left(s + \frac{1}{T}\right)}$$

$$\Rightarrow C(s) = \frac{1}{s} - \frac{1}{s + \frac{1}{T}}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\left(\frac{t}{T}\right)}\right) u(t)$$

Step Response of First Order System

The **unit step response**, c(t) has both the transient and the steady state terms.

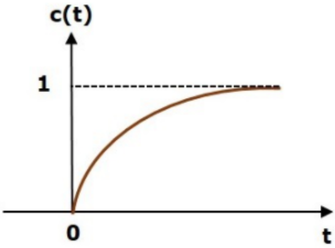
The transient term in the unit step response is -

$$c_{tr}(t) = -e^{-\left(\frac{t}{T}\right)}u(t)$$

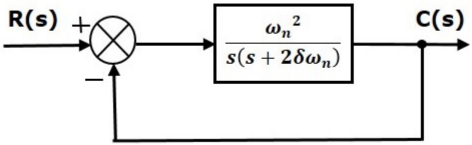
The steady state term in the unit step response is -

$$c_{ss}(t) = u(t)$$

The following figure shows the unit step response.



Response of Second Order System



Response of Second Order System

We know that the transfer function of the closed loop control system having unity negative feedback as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$

Substitute, $G(s) = \frac{\omega_n^2}{s(s+2\delta\omega_n)}$ in the above equation.

$$\frac{C(s)}{R(s)} = \frac{\left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)}{1 + \left(\frac{\omega_n^2}{s(s+2\delta\omega_n)}\right)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Response of Second Order System

The power of 's' is two in the denominator term. Hence, the above transfer function is of the second order and the system is said to be the **second order system**.

The characteristic equation is -

$$s^2 + 2\delta\omega_n s + \omega_n^2 = 0$$

The roots of characteristic equation are -

$$s = \frac{-2\omega\delta_n \pm \sqrt{(2\delta\omega_n)^2 - 4\omega_n^2}}{2} = \frac{-2(\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1})}{2}$$

$$\Rightarrow s = -\delta\omega_n \pm \omega_n\sqrt{\delta^2 - 1}$$

- The two roots are imaginary when $\delta = 0$.
- The two roots are real and equal when $\delta = 1$.
- The two roots are real but not equal when $\delta > 1$.
- The two roots are complex conjugate when $0 < \delta < 1$.

Response of Second Order System

We can write $C(s)$ equation as,

$$C(s) = \left(\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2} \right) R(s)$$

Where,

- **C(s)** is the Laplace transform of the output signal, c(t)
- **R(s)** is the Laplace transform of the input signal, r(t)
- $\boldsymbol{\omega_n}$ is the natural frequency
- $\boldsymbol{\delta}$ is the damping ratio.

Step Response of Second Order System

Consider the unit step signal as an input to the second order system.

Laplace transform of the unit step signal is,

$$R(s) = \frac{1}{s}$$

We know the transfer function of the second order closed loop control system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

Step Response of Second Order System

Case 1: δ = 0

Substitute, $\delta = 0$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Step Response of Second Order System

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{s^2 + \omega_n^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s^2 + \omega_n^2)}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - \cos(\omega_n t))\, u(t)$$

So, the unit step response of the second order system when $/delta = 0$ will be a continuous time signal with constant amplitude and frequency.

Step Response of Second Order System

Case 2: δ = 1

Substitute, $/delta = 1$ in the transfer function.

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\omega_n s + \omega_n^2}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right) R(s)$$

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\omega_n)^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\omega_n)^2}$$

Step Response of Second Order System

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)^2} = \frac{A}{s} + \frac{B}{s+\omega_n} + \frac{C}{(s+\omega_n)^2}$$

After simplifying, you will get the values of A, B and C as $1, -1\, and\, -\omega_n$ respectively.

Substitute these values in the above partial fraction expansion of $C(s)$.

$$C(s) = \frac{1}{s} - \frac{1}{s+\omega_n} - \frac{\omega_n}{(s+\omega_n)^2}$$

Apply inverse Laplace transform on both the sides.

$$c(t) = (1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t})u(t)$$

Step Response of Second Order System

Case 3: $0 < \delta < 1$

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}\right) R(s)$$

Step Response of Second Order System

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s\left((s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)\right)} = \frac{A}{s} + \frac{Bs + C}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

After simplifying, you will get the values of A, B and C as $1, -1\, and\, -2\delta\omega_n$ respectively.

Substitute these values in the above partial fraction expansion of C(s).

$$C(s) = \frac{1}{s} - \frac{s + 2\delta\omega_n}{(s+\delta\omega_n)^2 + \omega_n^2(1-\delta^2)}$$

Step Response of Second Order System

$$C(s) = \frac{1}{s} - \frac{s + \delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)} - \frac{\delta\omega_n}{(s + \delta\omega_n)^2 + \omega_n^2(1 - \delta^2)}$$

$$C(s) = \frac{1}{s} - \frac{(s+\delta\omega_n)}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2} - \frac{\delta}{\sqrt{1-\delta^2}}\left(\frac{\omega_n\sqrt{1-\delta^2}}{(s+\delta\omega_n)^2 + (\omega_n\sqrt{1-\delta^2})^2}\right)$$

Substitute, $\omega_n\sqrt{1-\delta^2}$ as ω_d in the above equation.

$$C(s) = \frac{1}{s} - \frac{(s + \delta\omega_n)}{(s + \delta\omega_n)^2 + \omega_d^2} - \frac{\delta}{\sqrt{1 - \delta^2}}\left(\frac{\omega_d}{(s + \delta\omega_n)^2 + \omega_d^2}\right)$$

Step Response of Second Order System

Apply inverse Laplace transform on both the sides.

$$c(t) = \left(1 - e^{-\delta\omega_n t}\cos(\omega_d t) - \frac{\delta}{\sqrt{1-\delta^2}} e^{-\delta\omega_n t}\sin(\omega_d t)\right) u(t)$$

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\left((\sqrt{1-\delta^2})\cos(\omega_d t) + \delta\sin(\omega_d t)\right)\right) u(t)$$

If $\sqrt{1-\delta^2} = \sin(\theta)$, then 'δ' will be cos(θ). Substitute these values in the above equation.

$$c(t) = \left(1 - \frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}(\sin(\theta)\cos(\omega_d t) + \cos(\theta)\sin(\omega_d t))\right) u(t)$$

$$\Rightarrow c(t) = \left(1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)\right) u(t)$$

So, the unit step response of the second order system is having damped oscillations (decreasing amplitude) when 'δ' lies between zero and one.

Step Response of Second Order System

Case 4: δ > 1

We can modify the denominator term of the transfer function as follows –

$$s^2 + 2\delta\omega_n s + \omega_n^2 = \left\{s^2 + 2(s)(\delta\omega_n) + (\delta\omega_n)^2\right\} + \omega_n^2 - (\delta\omega_n)^2$$

$$= (s + \delta\omega_n)^2 - \omega_n^2\left(\delta^2 - 1\right)$$

The transfer function becomes,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}$$

$$\Rightarrow C(s) = \left(\frac{\omega_n^2}{(s + \delta\omega_n)^2 - \omega_n^2(\delta^2 - 1)}\right) R(s)$$

Step Response of Second Order System

Substitute, $R(s) = \frac{1}{s}$ in the above equation.

$$C(s) = \left(\frac{\omega_n^2}{(s+\delta\omega_n)^2 - (\omega_n\sqrt{\delta^2-1})^2}\right)\left(\frac{1}{s}\right) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

Do partial fractions of $C(s)$.

$$C(s) = \frac{\omega_n^2}{s(s+\delta\omega_n+\omega_n\sqrt{\delta^2-1})(s+\delta\omega_n-\omega_n\sqrt{\delta^2-1})}$$

$$= \frac{A}{s} + \frac{B}{s+\delta\omega_n+\omega_n\sqrt{\delta^2-1}} + \frac{C}{s+\delta\omega_n-\omega_n\sqrt{\delta^2-1}}$$

After simplifying, you will get the values of A, B and C as 1, $\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ and $\frac{-1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}$ respectively. Substitute these values in above partial fraction expansion of

Step Response of Second Order System

$$C(s) = \frac{1}{s} + \frac{1}{2(\delta + \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\left(\frac{1}{s + \delta\omega_n + \omega_n\sqrt{\delta^2 - 1}}\right) - \left(\frac{1}{2(\delta - \sqrt{\delta^2 - 1})(\sqrt{\delta^2 - 1})}\right)\left(\frac{1}{s + \delta\omega_n - \omega_n\sqrt{\delta^2 - 1}}\right)$$

Apply inverse Laplace transform on both the sides.

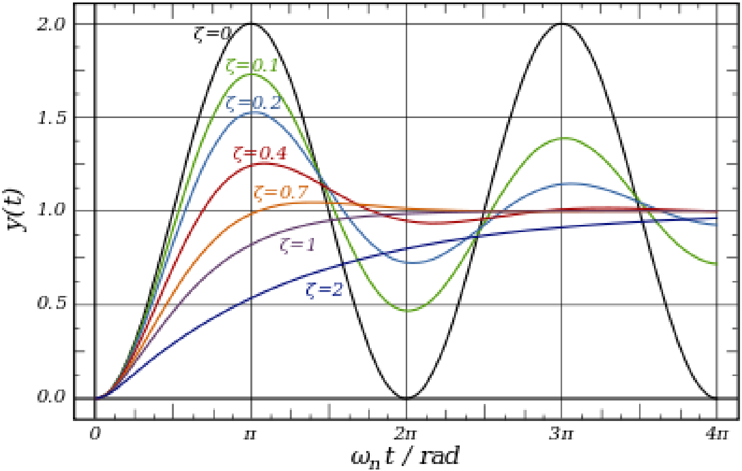
$$c(t) = \left(1 + \left(\frac{1}{2(\delta+\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n+\omega_n\sqrt{\delta^2-1})t} - \left(\frac{1}{2(\delta-\sqrt{\delta^2-1})(\sqrt{\delta^2-1})}\right)e^{-(\delta\omega_n-\omega_n\sqrt{\delta^2-1})t}\right)u(t)$$

Since it is over damped, the unit step response of the second order system when δ > 1 will never reach step input in the steady state.

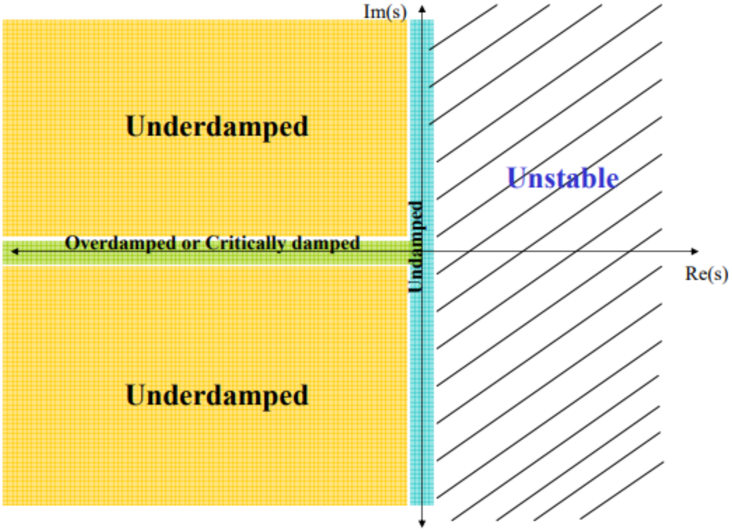
Damping

- Damping is an influence within or upon an oscillatory system that has the effect of reducing or preventing its oscillation.
- In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation.
- Examples include viscous drag in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators.
- The damping ratio is a dimensionless measure describing how oscillations in a system decay after a disturbance.

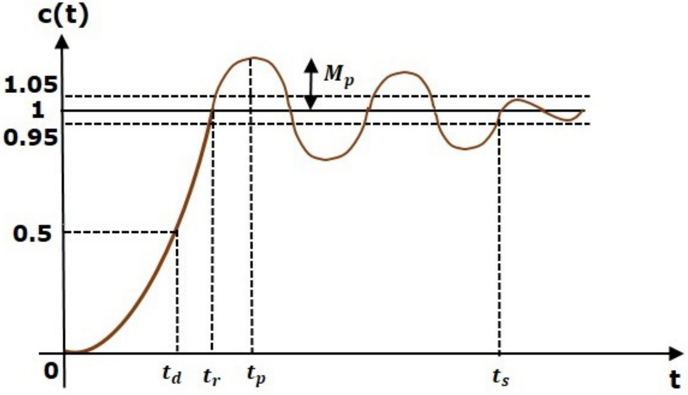
The effect of varying damping ratio on a second-order system.



Stability: Step Response of Second Order System



step response of the second order system for the underdamped case



The response up to the settling time is known as transient response and the response after the settling time is known as steady state response.

Delay Time t_d

- The time required for the response to reach 50% of the final value in the first time is called the delay time.

Delay Time t_d

Consider the step response of the second order system for $t \geq 0$, when 'δ' lies between zero and one.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

The final value of the step response is one.

Therefore, at $t = t_d$, the value of the step response will be 0.5. Substitute, these values in the above equation.

$$c(t_d) = 0.5 = 1 - \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_d}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as

$$t_d = \frac{1 + 0.7\delta}{\omega_n}$$

Rise Time t_r

- The time required for response to rising from 10% to 90% of final value, for an overdamped system and 0 to 100% for an underdamped system is called the rise time of the system.

Rise Time t_r

At t = t_1 = 0, c(t) = 0.

We know that the final value of the step response is one.

Therefore, at $t = t_2$, the value of step response is one. Substitute, these values in the following equation.

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

$$c(t_2) = 1 = 1 - \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta)$$

$$\Rightarrow \left(\frac{e^{-\delta\omega_n t_2}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \sin(\omega_d t_2 + \theta) = 0$$

$$\Rightarrow \omega_d t_2 + \theta = \pi$$

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Rise Time t_r

$$\Rightarrow t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of **rise time**,

$$t_r = t_2 - t_1$$

$$\therefore\ t_r = \frac{\pi - \theta}{\omega_d}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

Peak Time t_p

- It is the time required for the response to reach the peak value for the first time.

Peak Time t_p

t_p . At $t = t_p$, the first derivate of the response is zero.

We know the step response of second order system for under-damped case is

$$c(t) = 1 - \left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Differentiate $c(t)$ with respect to 't'.

$$\frac{dc(t)}{dt} = -\left(\frac{e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\omega_d\cos(\omega_d t + \theta) - \left(\frac{-\delta\omega_n e^{-\delta\omega_n t}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t + \theta)$$

Substitute, $t = t_p$ and $\frac{dc(t)}{dt} = 0$ in the above equation.

$$0 = -\left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\left[\omega_d\cos(\omega_d t_p + \theta) - \delta\omega_n\sin(\omega_d t_p + \theta)\right]$$

Peak Time t_p

$$\Rightarrow \omega_n \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta\omega_n \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sqrt{1-\delta^2}\cos(\omega_d t_p + \theta) - \delta \sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta)\cos(\omega_d t_p + \theta) - \cos(\theta)\sin(\omega_d t_p + \theta) = 0$$

$$\Rightarrow \sin(\theta - \omega_d t_p - \theta) = 0$$

$$\Rightarrow sin(-\omega_d t_p) = 0 \Rightarrow -\sin(\omega_d t_p) = 0 \Rightarrow sin(\omega_d t_p) = 0$$

$$\Rightarrow \omega_d t_p = \pi$$

$$\Rightarrow t_p = \frac{\pi}{\omega_d}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

Peak Overshoot M_p

- The difference between the peak of 1st time and steady output is called the peak overshoot. It is also called the maximum overshoot.

Mathematically, we can write it as

$$M_p = c(t_p) - c(\infty)$$

Where,

$c(t_p)$ is the peak value of the response.

$c(\infty)$ is the final (steady state) value of the response.

At $t = t_p$, the response c(t) is -

$$c(t_p) = 1 - \left(\frac{e^{-\delta\omega_n t_p}}{\sqrt{1-\delta^2}}\right)\sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right hand side of the above equation.

$$c(tp) = 1 - \left(\frac{e^{-\delta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\delta^2}}\right)\sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

$$\Rightarrow c(t_p) = 1 - \left(\frac{e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}}{\sqrt{1-\delta^2}}\right)(-\sin(\theta))$$

Peak Overshoot M_p

We know that

$$\sin(\theta) = \sqrt{1-\delta^2}$$

So, we will get $c(t_p)$ as

$$c(t_p) = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Substitute the values of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = 1 + e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} - 1$$

$$\Rightarrow M_p = e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}$$

Peak Overshoot M_p

Percentage of peak overshoot % M_p can be calculated by using this formula.

$$\%M_p = \frac{M_p}{c(\infty)} \times 100\%$$

By substituting the values of M_p and $c(\infty)$ in above formula, we will get the Percentage of the peak overshoot $\%M_p$ as

$$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)} \right) \times 100\%$$

From the above equation, we can conclude that the percentage of peak overshoot $\%M_p$ will decrease if the damping ratio δ increases.

Settling time t_s

- The time that is required for the response to reach and stay within the specified range (2% to 5%) of its final value is called the settling time.

Settling time t_s

The settling time for 5% tolerance band is -

$$t_s = \frac{3}{\delta\omega_n} = 3\tau$$

The settling time for 2% tolerance band is -

$$t_s = \frac{4}{\delta\omega_n} = 4\tau$$

Where, τ is the time constant and is equal to $\frac{1}{\delta\omega_n}$.

- Both the settling time t_s and the time constant τ are inversely proportional to the damping ratio δ .
- Both the settling time t_s and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time t_s and time constant τ will never change.

Example for Time Specifications

- Find the time domain specifications of a control system having the closed loop transfer function $\frac{4}{s^2+2s+4}$ when the unit step signal is applied as an input to this control system.

Example for Time Specifications

We know that the standard form of the transfer function of the second order closed loop control system as

$$\frac{\omega_n^2}{s^2 + 2\delta\omega_n s + \omega_n^2}$$

By equating these two transfer functions, we will get the un-damped natural frequency ω_n as 2 rad/sec and the damping ratio δ as 0.5.

We know the formula for damped frequency ω_d as

$$\omega_d = \omega_n\sqrt{1 - \delta^2}$$

Example for Time Specifications

Substitute, ω_n and δ values in the above formula.

$$\Rightarrow \omega_d = 2\sqrt{1-(0.5)^2}$$

$$\Rightarrow \omega_d = 1.732\ rad/sec$$

Substitute, δ value in following relation

$$\theta = \cos^{-1}\delta$$

$$\Rightarrow \theta = \cos^{-1}(0.5) = \frac{\pi}{3}\ rad$$

Example for Time Specifications

Time domain specification	Formula	Substitution of values in Formula	Final value
Delay time	$t_d = \frac{1+0.7\delta}{\omega_n}$	$t_d = \frac{1+0.7(0.5)}{2}$	t_d =0.675 sec
Rise time	$t_r = \frac{\pi-\theta}{\omega_d}$	$t_r = \frac{\pi-(\frac{\pi}{3})}{1.732}$	t_r =1.207 sec
Peak time	$t_p = \frac{\pi}{\omega_d}$	$t_p = \frac{\pi}{1.732}$	t_p =1.813 sec
% Peak overshoot	$\%M_p = \left(e^{-\left(\frac{\delta\pi}{\sqrt{1-\delta^2}}\right)}\right) \times 100\%$	$\%M_p = \left(e^{-\left(\frac{0.5\pi}{\sqrt{1-(0.5)^2}}\right)}\right) \times 100\%$	$\% M_p$ =16.32%
Settling time for 2% tolerance band	$t_s = \frac{4}{\delta\omega_n}$	$t_S = \frac{4}{(0.5)(2)}$	t_s =4 sec

Example for Time Specifications

Example 7.17.2 *For the system shown in the Fig. 7.17.4 obtain the closed loop T.F., damping ratio, natural frequency and expression for the output response if subjected to unit step input.*

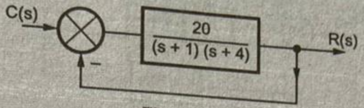


Fig. 7.17.4

Example for Time Specifications

Solution : $$\frac{C(s)}{R(s)} = \frac{\frac{20}{(s+1)(s+4)}}{1+\frac{20}{(s+1)(s+4)}} = \frac{20}{s^2+5s+24}$$

Key Point *Now though T.F. is not in standard form, denominator always reflect $2\xi\omega_n$ and ω_n^2 from middle term and the last term respectively.*

$\therefore$ Comparing, $s^2+5s+24$ with $s^2+2\xi\omega_n s+\omega_n^2$

$$\therefore \quad \omega_n^2 = 24 \quad \therefore \omega_n = \textbf{4.8989 rad/sec.}$$

$$2\xi\omega_n = 5 \quad \therefore \xi = \textbf{0.51031}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2} = \textbf{4.2129 rad/sec.}$$

Example for Time Specifications

Now, for c(t) we can use standard expression for $\frac{C(s)}{R(s)}$ in standard form. So writing

$$\frac{C(s)}{R(s)} = \frac{20}{24} \cdot \left\{ \frac{24}{s^2 + 5s + 24} \right\}$$

For the bracket term use standard expression, and then c(t) can be obtained by multiplying this expression by constant $\frac{20}{24}$.

$$\therefore \quad c(t) = \frac{20}{24}\left[1 - \frac{e^{-\xi \omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)\right]$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi} \text{ radians} = 1.03 \text{ radians}$$

$$\therefore \quad c(t) = \frac{20}{24}\left[1 - 1.1628\, e^{-2.5\, t} \sin(4.2129\, t + 1.03)\right]$$

Steady State Errors and Error constants

The deviation of the output of control system from desired response during steady state is known as **steady state error**. It is represented as e_{ss} . We can find steady state error using the final value theorem as follows.

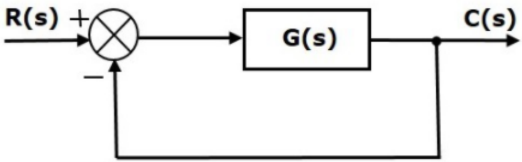
$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s)$$

Where,

E(s) is the Laplace transform of the error signal, $e(t)$

Steady State Errors for Unity Feedback Systems

Consider the following block diagram of closed loop control system, which is having unity negative feedback.



Where,

- R(s) is the Laplace transform of the reference Input signal $r(t)$
- C(s) is the Laplace transform of the output signal $c(t)$

Steady State Errors for Unity Feedback Systems

We know the transfer function of the unity negative feedback closed loop control system as

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)}$$

$$\Rightarrow C(s) = \frac{R(s)G(s)}{1+G(s)}$$

The output of the summing point is -

$$E(s) = R(s) - C(s)$$

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

Steady State Errors for Unity Feedback Systems

Substitute $C(s)$ value in the above equation.

$$E(s) = R(s) - \frac{R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s) + R(s)G(s) - R(s)G(s)}{1+G(s)}$$

$$\Rightarrow E(s) = \frac{R(s)}{1+G(s)}$$

Substitute $E(s)$ value in the steady state error formula

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)}$$

Steady State Errors and the error constants for standard input signals

Input signal	Steady state error e_{ss}	Error constant
unit step signal	$\frac{1}{1+k_p}$	$K_p = \lim_{s \to 0} G(s)$
unit ramp signal	$\frac{1}{K_v}$	$K_v = \lim_{s \to 0} sG(s)$
unit parabolic signal	$\frac{1}{K_a}$	$K_a = \lim_{s \to 0} s^2G(s)$

Where, K_p , K_v and K_a are position error constant, velocity error constant and acceleration error constant respectively.

Example Steady State Errors for Unity Feedback Systems

Example 7.8.1 *An unity feedback system has* $G(s) = \dfrac{20\,(1+s)}{s^2(2+s)(4+s)}$, *calculate its steady state error co-efficients and error when the applied input* $r(t) = 40 + 2t + 5t^2$.

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Example Steady State Errors for Unity Feedback Systems

Solution : $$G(s)H(s) = \frac{20(1+s)}{s^2(2+s)(4+s)} = \frac{20(1+s)}{s^2 \times 2\left(1+\frac{s}{2}\right)\times 4\left(1+\frac{s}{4}\right)}$$

$$\therefore \quad G(s)H(s) = \frac{2.5(1+s)}{s^2(1+0.5\,s)(1+0.25\,s)} \quad \text{...Sta}$$

Thus j = 2 hence it is Type 2 system

$$K_p = \lim_{s\to 0} G(s)H(s) = \lim_{s\to 0} \frac{2.5(1+s)}{s^2(1+0.5\,s)(1+0.25\,s)} = \infty$$

$$K_v = \lim_{s\to 0} sG(s)H(s) = \lim_{s\to 0} \frac{s\times 2.5(1+s)}{s^2(1+0.5\,s)(1+0.25\,s)} = \infty$$

Example Steady State Errors for Unity Feedback Systems

$$K_a = \lim_{s\to 0} s^2 G(s)H(s) = \lim_{s\to 0} \frac{s^2 \times 2.5\,(1+s)}{s^2\,(1+0.5\,s)\,(1+0.25\,s)} = 2.5$$

$$r(t) = 40 + 2t + 5t^2 = 40 + 2t + \frac{10}{2} t^2 = A_1 + A_2 t + \frac{A_3}{2} t^2$$

Hence $A_1 = 40$ step, $A_2 = 2$ ramp, $A_3 = 10$ parabolic

$$\therefore\ e_{ss} = e_{ss1} + e_{ss2} + e_{ss3} = \frac{A_1}{1+K_p} + \frac{A_2}{K_v} + \frac{A_3}{K_a}$$

$$\therefore\ e_{ss} = \frac{40}{1+\infty} + \frac{2}{\infty} + \frac{10}{2.5} = 0 + 0 + 4 = 4$$

The End